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On circle graphs with girth at least five

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Abstract

Circle graphs with girth at least five are known to be 2-degenerate (Ageev, 1999). In this paper, we prove that circle graphs with girth at least $g \geq 5$ contain a vertex of degree at most one or a chain of $g - 4$ vertices of degree two, which implies Ageev's result in the case $g = 5$. We then use this structural property to give an upper bound on the circular chromatic number of circle graphs with girth at least $g \geq 5$ as well as a precise estimate of their maximum average degree.

1   Introduction

Let C denote the unit circle, and let us take the clockwise orientation as the positive orientation of C. Let $\{x_0, \ldots, x_{k-1}\} \subset C$, we say that $(x_0, \ldots, x_{k-1})$ are in *cyclic order* if the minimum between the sum of the length of the arcs $\overrightarrow{x_i x_{i+1}}$, $0 \leq i \leq k - 1$, and the sum of the length of the arcs $\overrightarrow{x_{i+1} x_i}$, $0 \leq i \leq k - 1$, is equal to one, where i is taken modulo k. A pair $\{x, y\}$ of elements of C is called a *chord* of C with *endpoints* x and y. Two chords $\{x_1, y_1\}$ and $\{x_2, y_2\}$ *intersect* if $(x_1 x_2 y_1 y_2)$ are in cyclic order, otherwise they are said to be *parallel*.

All graphs considered in this paper are simple: they do not have any loop nor parallel edges. The *girth* of a graph G is the size of a shortest cycle in G. We call a k-vertex (resp. $^{\leq}k$-vertex, $^{\geq}k$-vertex) a vertex of degree k (resp. at most k, at least k).

By definition, every circle graph G with set of vertices $V(G) = \{v_1, \ldots, v_n\}$ admits a *representation* $\mathcal{C} = \{\{x_1, y_1\}, \ldots, \{x_n, y_n\}\}$ such that for all i, j, v_i and v_j are adjacent in G if and only if the chords $\{x_i, y_i\}$ and $\{x_j, y_j\}$
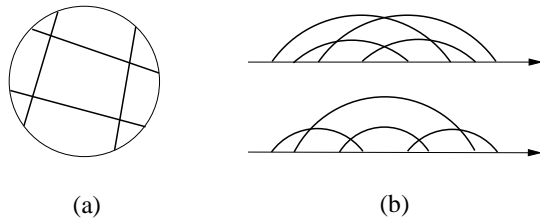
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Figure 1: (a) The unique circle representation of C_4. (b) The two non-equivalent representations of C_4 on the real axis.

intersect in $\mathcal{C}$. We only consider representations in which endpoints and intersection points of chords are all distinct. Observe that in general, circle graphs do not have a unique representation. A representation $\mathcal{C}'$ obtained from $\mathcal{C}$ only by removing chords is called a *sub-representation* of $\mathcal{C}$. Observe that if $\mathcal{C}$ is a representation of G, a sub-representation of $\mathcal{C}$ corresponds to an induced subgraph of G.

Observation 1 *Let G be a circle graph with representation $\mathcal{C}$, and let $v_1, \ldots, v_k$ be an independent set in G. The chords of $\mathcal{C}$ corresponding to $v_1, \ldots, v_k$ are pairwise parallel.*

In order to prove that circle graphs with girth at least five are 2-degenerate, Ageev [1] does not consider their circle representation, but an equivalent representation on the real axis, usually called *interval-overlap*. The major difference is that some graphs, for example cycles, have a unique circle representation whereas they have several non-equivalent representations on the real axis (see Figure 1). Hence, even if considering a real axis representation can be very convenient to define an order on the endpoint of the chords, the case study is then much harder. Unfortunately, even in the circle representation, some very simple graphs such as the union of two disjoint paths do not have a unique representation (see Figure 2). Observe that in Figure 2(a), the representation of the two paths is a sub-representation of the representation of a cycle. In this case we make a slight abuse of notation and say that the two paths are *in cyclic order*.

In Section 2, we prove the following extension of Ageev's result:

Theorem 1 *Every circle graph with girth $g \geq 5$ contains a $^{\leq 1}$-vertex or a chain of $(g-4)$ 2-vertices.*
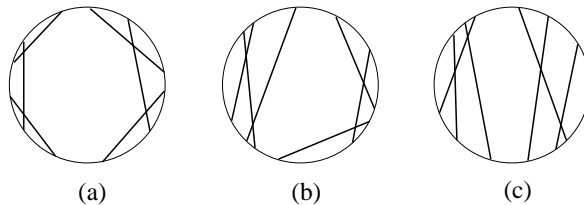
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Figure 2: Three non-equivalent circle representations of the union of two paths of length two.

In [1], Ageev uses his structural result to prove that circle graphs with girth at least five have chromatic number at most three. We can use Theorem 1 to obtain a refinement of this result for circle graphs with larger girth. Instead of considering the chromatic number of these graphs, we consider their circular chromatic number. For two integers $1 \leq q \leq p$, a (p,q)-coloring of a graph G is a coloring c of the vertices of G with colors $\{0, \ldots, p-1\}$ such that for any pair of adjacent vertices x and y, we have $q \leq |c(x) - c(y)| \leq p - q$. The circular chromatic number of G is

$$\chi_c(G) = \inf \left(\frac{p}{q} \mid \text{ there exists a } (p,q)\text{-coloring of } G \right).$$

It is known that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, and so $\chi(G) = \lceil \chi_c(G) \rceil$. The chromatic number can thus be considered as an approximation of the circular chromatic number.

Using a well-known observation on circular coloring (see e.g. Corollary 2.2 in [2]), the existence of a chain of $(g-4)$ 2-vertices implies the following result:

Corollary 1 *Every circle graph G with girth $g \geq 5$ has circular chromatic number*

$$\chi_c(G) \leq 2 + \frac{1}{\left\lceil \frac{g-3}{2} \right\rceil}.$$

In Section 3, we study an invariant giving a very precise idea of the local structure of graphs. The *maximum average degree* of a graph G is defined as

$$\mathrm{mad}(G) = \max \left\{ \mathrm{ad}(H), H \subseteq G \right\}, \text{ where } \mathrm{ad}(H) = \frac{2|E(H)|}{|V(H)|}.$$

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Class	Planar	Outerplanar	Partial 2-Tree	Seg	1-String
μ_g	$2 + \frac{4}{g-2}$	$2 + \frac{2}{g-2}$	$2 + \frac{2}{\left\lceil \frac{g-1}{2} \right\rceil}$	$2 + \frac{4}{g-4}$	$2 + \frac{4}{g-4}$

Table 1: Values of μ_g for some classes of graphs.

For planar graphs, there is a simple relation between girth and maximum average degree: any planar graph G with girth g is such that $\mathrm{mad}(G) < 2g/(g-2)$. On the other hand, there exists a family $(G_n)_{n \geq 0}$ of planar graphs with girth g, such that $\mathrm{mad}(G_n) \to 2g/(g-2)$ when $n \to \infty$. We would like to obtain the same kind of link between the girth and the maximum average degree of circle graphs. The following corollary is a straightforward consequence of Theorem 1:

Corollary 2 *Any circle graph G with girth $g \geq 5$ is such that $\mathrm{mad}(G) < 2 + 2/(g-4)$.*

note that Corollary 2 has some implications on the circular choosability of circle graphs. Using Proposition 32(i) in Section 5.4 of [3], we can prove :

Corollary 3 *Every circle graph G with girth $g \geq 5$ has circular choice number $cch(G) \leq 2 + \frac{4}{g-2}$.*

To improve Corollary 2, we consider

$$\mu_g(\mathcal{F}) = \sup \{\mathrm{mad}(G) \mid G \in \mathcal{F} \text{ and } G \text{ has girth at least } g\}.$$

Let Seg denote the class of graphs defined as intersection of segments in the plane, and 1-String denote the class of graphs defined as intersection of jordan curves in the plane, such that any two curves intersect at most once. Table 1 gives an idea of the function μ_g for some classes of graphs. Note that for Seg and 1-String, g has to be at least five, since otherwise μ_g is not bounded.

We can remark that for all these classes, μ_g is a rational number. The following theorem shows that this is not the case for the class of circle graphs. It is proved in Section 3.

Theorem 2 *For every $g \geq 5$, $\mu_g(\mathrm{CIRCLE}) = 2\sqrt{\frac{g-2}{g-4}}$*

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2 Proof of Theorem 1

Let $G = (V, E)$ be a circle graph with girth $g \geq 5$ and minimum degree two, and let $\mathcal{C} = \{\{x_1, x_1'\}, \ldots, \{x_n, x_n'\}\}$ be a circle representation of G. We first decompose the chords of $\mathcal{C}$ into two sets, using the following rules:

(1) for every set of 3 distincts chords $\{x, x'\}$, $\{y, y'\}$, and $\{z, z'\}$, such that $\{y, y'\}$ is uncolored and $(xyzz'y'x')$ are in cyclic order, colour the chord $\{y, y'\}$ in blue,

(2) colour all the uncolored chords in red.

By construction, the red chords are exactly the chords $\{x, y\}$ such that at least one of the arcs $\overrightarrow{xy}$ and $\overrightarrow{yx}$ does not contain both endpoints of a chord distinct from $\{x, y\}$. Let $\mathcal{C}^R$ (resp. $\mathcal{C}^B$) be the representation induced by the red (resp. blue) chords and G^R (resp. G^B) be the corresponding graph. We first prove the following lemma.

Lemma 1 $\mathcal{C}^R$ *is a sub-representation of the representation of a cycle.*

Proof. Assume that G^R contains a ≥ 3-vertex v, adjacent to x, y, and z in G^R. Since $g \geq 5$, the graph G does not contain any triangle, and so $\{x, y, z\}$ is an independent set. Using Observation 1, this implies that the three corresponding red chords are parallel in any representation, which contradicts Rule (1).

Hence, G^R has maximum degree two. Suppose now that G^R contains a cycle. Then if there exists a vertex which is not in the cycle, the corresponding chord, as well the chords corresponding to two non-adjacent vertices of the cycle, are parallel (recall that the cycle has length at least five, since $g \geq 5$). This contradicts Rule (1). So G^R is either a cycle or a union of disjoint paths.

Suppose now that $\mathcal{C}^R$ is not a sub-representation of a cycle. Then G^R is necessarily a union of disjoint paths, and two of them are not in cyclic order in $\mathcal{C}^R$. This also contradicts Rule (1), so $\mathcal{C}^R$ is a sub-representation of the representation of a cycle. $\qquad\square$

Observe that each blue chord $\{x, y\}$ induces two complementary arcs $\overrightarrow{xy}$ and $\overrightarrow{yx}$ on the circle. We denote by $\mathcal{A}_1$ the set of such arcs. Similarly, two intersecting blue chords $\{u, v\}$ and $\{x, y\}$ induce four consecutive arcs whose lengths add up to one, say without loss of generality $\overrightarrow{ux}$, $\overrightarrow{xv}$, $\overrightarrow{vy}$, and $\overrightarrow{yu}$. We denote by $\mathcal{A}_2$ the set of all such arcs.

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For any arc $\overrightarrow{xy}$ of the circle, we define $\rho(\overrightarrow{xy})$ as the number of red chords having both endpoints in $\overrightarrow{xy}$. We consider the integer $t = \min\{\rho(\overrightarrow{xy}), \overrightarrow{xy} \in \mathcal{A}_1 \cup \mathcal{A}_2, \rho(\overrightarrow{xy}) > 0\}$.

If there is no blue chord in our decomposition, then G is either a cycle or a union of paths, and thus contains a $^{\leq}1$-vertex or g adjacent 2-vertices. So we can assume from now on that G^B is non empty. Observe that for any blue chord $\{x, y\}$, we have $\rho(\overrightarrow{xy}) > 0$ and $\rho(\overrightarrow{yx}) > 0$ since otherwise $\{x, y\}$ would be red. Hence, the integer t exists. We now consider two cases, depending on whether the minimum is reached by two intersecting chords or by a single chord.

Case 1: The minimum $t > 0$ is reached by two intersecting blue chords, say $\{x, x'\}$ and $\{y, y'\}$, and for every blue chord $\{u, v\}$, we have $\rho(\overrightarrow{uv}) \neq t$. Let us assume without loss of generality that $t = \rho(\overrightarrow{xy})$. According to the clockwise order, we denote by $\{x_1, x_1'\}, \ldots \{x_t, x_t'\}$ the red chords having both endpoints in $\overrightarrow{xy}$ (see Figure 3(a)). Observe that every blue chord has at most one endpoint in $\overrightarrow{xy}$, since otherwise we would have a blue chord $\{u, v\}$ with $1 \leq \rho(\overrightarrow{uv}) \leq t$, which would contradict the hypothesis.

We first prove that the graph induced by the chords $\{x_i, x_i'\}$ ($1 \leq i \leq t$) is a path. If this is not the case, then for some i the chords $\{x_i, x_i'\}$ and $\{x_{i+1}, x_{i+1}'\}$ do not intersect. Then either one of them corresponds to a $^{\leq}1$-vertex, or each of them intersects a blue chord. Such a blue chord also intersects $\{x, x'\}$ or $\{y, y'\}$, since it has only one endpoint in $\overrightarrow{xy}$. This contradicts the minimality of t.

We now prove that the arc $\overrightarrow{x_2 x_{t-1}'}$ does not contain any endpoint of a blue chord. Observe that if the arc contains the endpoint u of a blue chord, then there exists $1 \leq i \leq t - 2$ such that $u \in \overrightarrow{x_i' x_{i+2}}$, since otherwise this would create a triangle. If such an endpoint u exists, the related blue chord along with $\{x, x'\}$ or $\{y, y'\}$ contradicts the minimality of t.

Hence, the vertices corresponding to $\{x_i, x_i'\}$ ($2 \leq i \leq t - 1$) are a chain of $(t - 2)$ 2-vertices in G. Since G does not contain any 1-vertex, the chord $\{x_1, x_1'\}$ intersects a chord $\{u, u'\}$ distinct from $\{x_2, x_2'\}$. Such a chord may be blue or red, but by the minimality of t it cannot intersect $\{y, y'\}$. So the chord $\{u, u'\}$ has to intersect $\{x, x'\}$ and since $g \geq 4$, exactly one such $\{u, u'\}$ exists. Similarly, $\{x_t, x_t'\}$ intersects exactly one chord distinct from $\{x_{t-1}, x_{t-1}'\}$, say $\{v, v'\}$, and $\{v, v'\}$ also intersects $\{y, y'\}$. Thus the vertices corresponding to $\{x_i, x_i'\}$ ($1 \leq i \leq t$) form a chain of t 2-vertices in G. Since the chords $\{x, x'\}, \{u, u'\}, \{x_1, x_1'\}, \ldots, \{x_t, x_t'\}, \{v, v'\}, \{y, y'\}$ correspond to
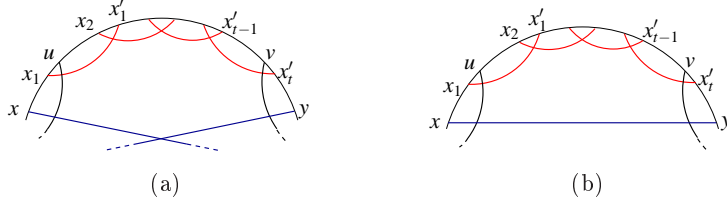
(a)   (b)

Figure 3: A chain of $t \geq g - 4$ vertices of degree two in G.

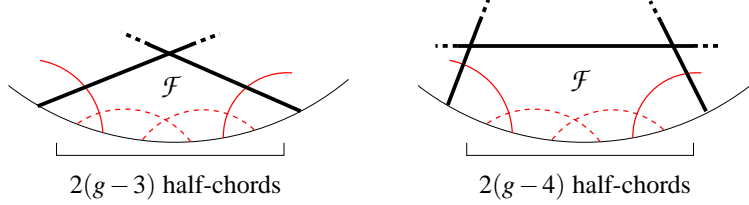$2(g-3)$ half-chords $2(g-4)$ half-chords

Figure 4: From $Q_{g,t}$ to $Q_{g,t+1}$

a cycle in G, we have $t \geq g - 4$.

Case 2: The minimum $t > 0$ is reached by a blue chord $\{x, y\}$. The proof is the same as the previous one, except that we obtain a chain of $(g - 3)$ 2-vertices instead of $(g - 4)$ 2-vertices (see Figure 3(b)).

3   Proof of Theorem 2

Let us first give a construction to prove the lower bound. For every $g \geq 5$, we construct a family $(Q_{g,t})_{t \geq 0}$ of circle graphs with girth g such that $Q_{g,0} = C_g$ (the cycle on n vertices) and $Q_{g,t+1}$ is obtained by adding chords to the representation of $Q_{g,t}$.

These new chords (represented as thin chords in Figure 4) induce a cycle. Every old chord (i.e. that belongs to $Q_{g,t}$, represented as thick chords in Figure 4) intersects one new chord at each of its endpoints. A *k-region* is a region inside the circle, which is incident to the circle and to exactly k chords. Note that in any $Q_{g,t}$, every k-region is either a 2- or a 3-region. Any 2-region in $Q_{g,t}$ produces in $Q_{g,t+1}$ a face $\mathcal{F}$ of size g, $(g - 3)$ vertices $(2(g - 3)$ half-chords), $(g - 2)$ edges, $(g - 3)$ 2-regions, and $(g - 2)$ 3-regions. Any 3-region in $Q_{g,t}$ produces in $Q_{g,t+1}$ a face $\mathcal{F}$ of size g, $(g - 4)$ vertices,
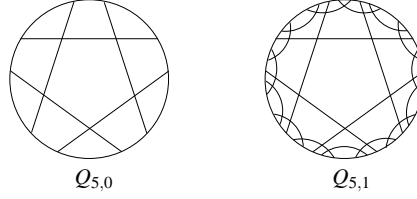
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$Q_{5,0}$ $Q_{5,1}$

Figure 5: Examples

$(g-3)$ edges, $(g-4)$ 2-regions, and $(g-3)$ 3-regions.

We now consider the vector $V_{g,t} = {}^t(n, m, R_2, R_3)$ whose components are respectively the number of vertices, edges, 2-regions, and 3-regions of $Q_{g,t}$. By construction, we have that $V_{g,t+1} = M_g V_{g,t}$, where

$$M_g = \begin{pmatrix} 1 & 0 & g-3 & g-4 \\ 0 & 1 & g-2 & g-3 \\ 0 & 0 & g-3 & g-4 \\ 0 & 0 & g-2 & g-3 \end{pmatrix}$$

The limit of the average degree $\mathrm{ad}(Q_{g,t})$ of $Q_{g,t}$ when $t \to \infty$ can be obtained from the unique eigenvector

$$V = \begin{pmatrix} g-3+\sqrt{(g-2)(g-4)} \\ g-2+(g-3)\sqrt{(g-2)/(g-4)} \\ g-4+\sqrt{(g-2)(g-4)} \\ g-2+\sqrt{(g-2)(g-4)} \end{pmatrix}$$

associated to the largest eigenvalue $g-3+\sqrt{(g-2)(g-4)}$ of M_g. We thus obtain:

$$\mu_g \geq \lim_{t\to\infty} \mathrm{ad}(Q_{g,t}) = 2 \cdot \frac{g-2+(g-3)\sqrt{(g-2)/(g-4)}}{g-3+\sqrt{(g-2)(g-4)}} = 2\sqrt{\frac{g-2}{g-4}}$$

Before proving the upper bound, we make some remarks on structure of the graphs $Q_{g,t}$. Observe that the graphs $Q_{g,t}$ with $t \geq 1$ are circle graphs with girth $g \geq 5$ that contain neither ${}^{\leq}1$-vertices nor chains of $(g-3)$ 2-vertices (see Figure 5 for an example with $g = 5$), which proves that Theorem 1 is optimal in a certain way. Another interesting property of these graphs is that for any $g \geq 5$, $Q_{g,t}$ contains K_{t+3}, the complete graph with $t+3$ vertices, as a minor (that is, K_{t+3} can be obtained from $Q_{g,t}$ by contracting edges and removing edges and vertices). To see this, contract $Q_{g,0}$

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in order to obtain a triangle, and at each step contract the set of new vertices into a single vertex, which is universal by construction. The size of the clique we construct will increase by one at each step, and we will eventually obtain K_{t+3} as a minor of $Q_{g,t}$. This implies that for any integer $g \geq 5$ and any graph H, there exists a circle graph G with girth g such that G contains H as a minor.

We now prove the upper bound by contradiction. Since circle graphs of girth at least g are closed under taking induced subgraphs, it is sufficient to prove that every circle graph G with girth at least $g \geq 5$ has average degree $\mathrm{ad}(G) < 2\sqrt{\frac{g-2}{g-4}}$.

Let G be a circle graph and $\mathcal{C}$ be a circle representation of G. We denote by $R(\mathcal{C})$ the planar graph constructed as follows:

- the vertex set of $R(\mathcal{C})$ is the set of crossings of chords in $\mathcal{C}$,

- two distinct vertices are adjacent in $R(\mathcal{C})$ if and only if they correspond to consecutive crossings of a same chord in $\mathcal{C}$.

Observe that the construction above clearly gives a natural planar embedding of $R(\mathcal{C})$. In the following, we only consider this precise planar embedding. For example, the outerface of $R(\mathcal{C})$ will be well-defined. Note that $R(\mathcal{C})$ has maximum degree four.

Let us consider a fixed integer $g \geq 5$ and a circle graph G_1 with girth at least g, such that $\mathrm{ad}(G_1) > 2\sqrt{\frac{g-2}{g-4}}$, and such that G_1 is minimal with this property. That is, for any circle graph H with girth at least g and such that $|V(H)| < |V(G_1)|$, we have $\mathrm{ad}(H) < 2\sqrt{\frac{g-2}{g-4}}$. Observe that by minimality, G_1 does not contain any $^{\leq}1$-vertex, since otherwise by removing it we would obtain a smaller graph with larger average degree.

Let $\mathcal{C}_1$ be a representation of G_1. If the outerface of the planar embedding of $R(\mathcal{C}_1)$ contains a 4-vertex, we apply the following operation on $\mathcal{C}_1$, which gives a new representation $\mathcal{C}_2$ and a new circle graph G_2 with girth g. Let u denote a 4-vertex on the outerface of $R(\mathcal{C}_1)$. It corresponds to an edge between to vertices v_1 and v_2 of G_1, represented by two crossing chords c_1 and c_2 in $\mathcal{C}_1$. Since u is a 4-vertex in $R(\mathcal{C}_1)$, the chords c_1 and c_2 respectively cross two chords c_1' and c_2' as depicted in Figure 6. Let v_1' and v_2' be the vertices of G_1 associated to c_1' and c_2'. Since u is on the outerface of $R(\mathcal{C}_1)$, v_1' and v_2' are not adjacent in G_1. Hence, we can add a path of $g - 4$ chords between c_1' and c_2', as depicted in Figure 6. Let $\mathcal{C}_2$ denote the new representation, and G_2 be the associated circle graph. The $g - 4$ vertices
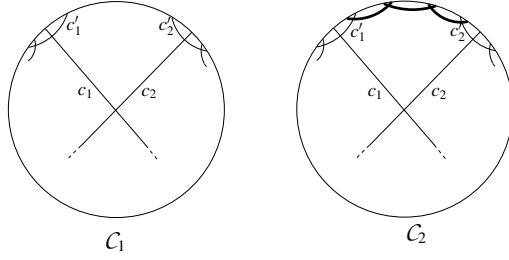
Figure 6: From $\mathcal{C}_1$ to $\mathcal{C}_2$

added to G_1 to obtain G_2 form a cycle of length exactly g in G_2 containing v_1, v_2, v_1', and v_2'. Note that the number of 4-vertices on the outerface of the plane graph associated to the representation decreases by one after at most two iterations of this process.

Let n_1 and m_1 denote respectively the number of vertices and edges of G_1. By Corollary 2, we have that $\mathrm{ad}(G_1) < 2 \cdot \frac{g-3}{g-4}$. This implies that $\mathrm{ad}(G_2) = 2 \cdot \frac{m_1+g-3}{n_1+g-4} > 2 \cdot \frac{m_1}{n_1} = \mathrm{ad}(G_1)$. Thus the average degree increases during this operation.

We repeat this operation until we obtain a circle graph G with girth g having a representation $\mathcal{C}$ such that the outerface of the planar embedding of $R(\mathcal{C})$ does not contain any 4-vertex. The consequence of the previous observation is that $\mathrm{ad}(G) > \mathrm{ad}(G_1) > 2\sqrt{\frac{g-2}{g-4}}$. Let n and m be the number of vertices and edges of G. This implies in particular that:

$$\sqrt{\frac{g-2}{g-4}} n < m \tag{1}$$

Let N, M, and F denote respectively the number of vertices, edges, and faces of $R(\mathcal{C})$. Since a crossing in $\mathcal{C}$ corresponds to both an edge in G and a vertex in $R(\mathcal{C})$, we have:

$$N = m \tag{2}$$

We can write Euler's formula for the planar embedding of $R(\mathcal{C})$ as follows:

$$M + 2 = F + N \tag{3}$$

Let N_d denote the number of d-vertices in $R(\mathcal{C})$. Since G_1 does not contain any $^{\leq}1$-vertex, and no new $^{\leq}1$-vertex is created during the transformation, the graph G does not contain any $^{\leq}1$-vertex either. This implies

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in particular that $R(\mathcal{C})$ does not contain $^{\leq}1$-vertices. Thus, the degree of a vertex in $R(\mathcal{C})$ is at least 2 and at most 4 and we have:

$$N = N_2 + N_3 + N_4 \tag{4}$$

The sum of vertex degrees is equal to twice the number of edges in $R(\mathcal{C})$:

$$2N_2 + 3N_3 + 4N_4 = 2M \tag{5}$$

Any chord in a representation of G corresponding to some vertex $v \in G$ contains $(deg(v) - 1)$ edges of $R(\mathcal{C})$. Since $\sum_{v \in G}(deg(v) - 1) = 2m - n$, we have:

$$2m - n = M \tag{6}$$

Note that the outerface of $R(\mathcal{C})$ contains every 2-vertex, every 3-vertex, and no 4-vertex of $R(\mathcal{C})$. Moreover, $R(\mathcal{C})$ cannot contain a face of degree strictly less than g, since otherwise G would contain a cycle of length strictly less than g. We thus obtain a lower bound on the sum of degrees of the faces of $R(\mathcal{C})$, which is equal to twice the number of edges in $R(\mathcal{C})$:

$$g(F - 1) + N_2 + N_3 \leq 2M \tag{7}$$

Let us decompose the chords of $\mathcal{C}$ into blue and red chords as done in the proof of Theorem 1. Using previous notation, $\mathcal{C}^B$ is the sub-representation of $\mathcal{C}$ induced by the blue chords and G^B is the corresponding circle graph. Note that G^B is a proper induced subgraph of G_1 and G. We thus have:

$$\mathrm{ad}\left(G^B\right) = \frac{2(m - N_2 - N_3)}{n - N_2} < 2\sqrt{\frac{g-2}{g-4}} < \frac{2m}{n} = \mathrm{ad}\left(G\right)$$

This implies that $\frac{2(N_2+N_3)}{N_2} > \frac{2m}{n} > 2\sqrt{\frac{g-2}{g-4}}$, which gives:

$$\left(\sqrt{\frac{g-2}{g-4}} - 1\right) N_2 < N_3 \tag{8}$$

The combination $(g-4) \times (\mathbf{1}) + (g-4)\left(2\sqrt{\frac{g-2}{g-4}} - 1\right) \times (\mathbf{2}) + g \times (\mathbf{3}) + 2(g-2)\left(1 - \sqrt{\frac{g-4}{g-2}}\right) \times (\mathbf{4}) + \frac{1}{2}(g-2)\left(1 - \sqrt{\frac{g-4}{g-2}}\right) \times (\mathbf{5}) + \sqrt{(g-2)(g-4)} \times (\mathbf{6}) + (\mathbf{7}) + \frac{1}{2}(g-4)\left(\sqrt{\frac{g-2}{g-4}} - 1\right) \times (\mathbf{8})$ gives $g < 0$, a contradiction.

4 Perspectives

In the present paper, we study the structure of sparse circle graphs. The opposite problem of studying the structure of dense circle graphs seems to be much harder. For example, the relation between the clique number of circle graphs and their chromatic number is not precisely established. Kostochka and Kratochvíl [4] proved that every circle graph with clique number ω has chromatic number at most $2^{\omega+6}$, but this is still far from the lower bound of $\Omega(\omega \log \omega)$.

Note that the upper bound of $2^{\omega+6}$ even holds for polygon-circle graphs, a superclass of circle graphs, defined as the intersection class of chords and convex polygons of the circle. The size of this class is known to be much larger, but we suspect that polygon-circle graphs with girth at least five behave like circle graphs with girth at least five. It would be interesting to see if the results of the present paper extend to the class of polygon-circle graphs.

Acknowledgement

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