Boxicity of graphs with bounded degree

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March 20, 2008

Abstract

The boxicity of a graph $G = (V, E)$ is the smallest k for which there exist k interval graphs $G_i = (V, E_i)$, $1 \le i \le k$, such that $E = E_1 \cap \ldots \cap E_k$. Graphs with boxicity at most d are exactly the intersection graphs of (axis-parallel) boxes in $\mathbb{R}^d$. In this note, we prove that graphs with maximum degree Δ have boxicity at most $\Delta^2 + 2$, which improves the previous bound of $2\Delta^2$ obtained by Chandran et al (J. Combin. Theory Ser. B **98** (2008) 443–445).

Keywords: Boxicity, interval graphs, bounded degree.

For a family $\mathcal{F} = \{S_1, \ldots, S_n\}$ of subsets of a set Ω, the *intersection graph* of $\mathcal{F}$ is defined as the graph with vertex set $\mathcal{F}$, in which two sets are adjacent if and only if their intersection is non-empty. A *d-box* is the cartesian product $[x_1, y_1] \times \ldots \times [x_d, y_d]$ of d closed intervals of the real line. For any graph G, the *boxicity* of G, denoted by $\text{box}(G)$, is the smallest d such that G is the intersection graph of a family of d-boxes.

For a family of graphs $\{G_i = (V, E_i),\ 1 \le i \le k\}$ defined on the same vertex set, we set $G_1 \cap \ldots \cap G_k$ to be the graph with vertex set V, and edge set $E_1 \cap \ldots \cap E_k$), and we naturally say that the graph $G_1 \cap \ldots \cap G_k$ is the intersection of the graphs $G_1, \ldots, G_k$. The boxicity of a graph G can be equivalently defined as the smallest k such that G is the intersection of k interval graphs. Graphs with boxicity one are exactly interval graphs, which can be recognized in linear time. On the other hand, Kratochvíl [5] proved that determining whether $\text{box}(G) \le 2$ is NP-complete.

The concept of boxicity was introduced in 1969 by Roberts [8]. It is used as a measure of the complexity of ecological [9] and social [4] networks, and has applications in fleet maintenance [7]. Boxicity has been investigated for various classes of graphs

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[3, 10, 11], and has been related with other parameters, such as treewidth [1]. Recently, Chandran et al. [2] proved that every graph with maximum degree at most Δ has boxicity at most $2\Delta^2$. To prove this bound, Chandran et al. use the fact that if a graph G is the intersection of k graphs $G_1, \ldots, G_k$, we have $\mathrm{box}(G) \leq \sum_{1 \leq i \leq k} \mathrm{box}(G_i)$.

In the remaining of the article, we use the same idea to prove the following theorem:

Theorem 1 *Every graph with maximum degree Δ has boxicity at most $2\lfloor \Delta^2/2 \rfloor + 2$.*

Proof. Let $G = (V, E)$ be a graph with maximum degree Δ, and let c be a (not necessarily proper) coloring of the vertices of G with colors from $\{1, \ldots, 2k\}$ such that:

(i)   there is no path uvw with $c(u) = c(w)$;

(ii) for any $1 \leq j \leq k$, there is no edge between a vertex colored with $2j - 1$ and a vertex colored with $2j$.

Observe that condition (i) implies that the graph induced by each color class is a graph with maximum degree at most one (the disjoint union of a stable set and a matching). The first step of the proof is to find the smallest k such that a $2k$-coloring as defined above exists. Define the function f such that for every $j \geq 1$, $f(2j) = 2j - 1$ and $f(2j - 1) = 2j$. We color the vertices of G one by one with the following procedure: while coloring a vertex $u \in V$, we choose for u a color from $\{1, \ldots, 2k\} \setminus (N_1 \cup N_2)$, where $N_1 = \{f(c(v)) \mid v$ is a colored neighbor of $u\}$ and $N_2 = \{c(v) \mid u$ and v have a common (not necessarily colored) neighbor$\}$.

If we follow this procedure, the partial coloring obtained at the end of each step has the desired properties : since $c(u) \notin N_1$, condition (ii) is still verified, and since $c(u) \notin N_2$, condition (i) is also still verified. At each step, N_1 has size at most Δ and N_2 has size at most $\Delta(\Delta - 1)$. Hence if $k = \left\lceil \frac{\Delta^2 + 1}{2} \right\rceil = \left\lfloor \frac{\Delta^2}{2} \right\rfloor + 1$, a $2k$-coloring of G as defined above exists.

From now on, we assume that $k = \lfloor \Delta^2/2 \rfloor + 1$. Hence, a $2k$-coloring c of G with the properties defined above exists. For any $1 \leq i \leq k$, let G_i be the graph obtained from G by adding an edge between any two non-adjacent vertices u, v such that $c(u), c(v) \notin \{2i - 1, 2i\}$. Using conditions (i) and (ii), G_i can be decomposed into a clique K_i (induced by the vertices colored neither with $2i - 1$, nor with $2i$), and two sets S_{2i-1} and S_{2i} corresponding to the vertices colored with $2i - 1$ and $2i$ respectively (see Figure 2(a)). By condition (ii), there is no edge between S_{2i-1} and S_{2i}, and by condition (i), every vertex of K_i is adjacent to at most one vertex of S_{2i-1} and one vertex of S_{2i}. Moreover, S_{2i-1} and S_{2i} both induce a graph with maximum degree one by condition (i).

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Now observe that $G = \cap_{1 \leq i \leq k} G_i$. If two vertices are adjacent in G they are also adjacent in any G_i, since $G \subseteq G_i$. On the other hand, if two vertices u and v are not adjacent in G, then they are not adjacent in $G_{\lceil c(u)/2 \rceil}$, and so they are not adjacent in the intersection of the G_i's.

As a consequence, $\mathrm{box}(G) \leq \sum_{1 \leq i \leq k} \mathrm{box}(G_i)$. We now show that every graph G_i has boxity at most two, which implies that $\mathrm{box}(G) \leq 2(\lfloor \Delta^2/2 \rfloor + 1)$ and concludes the proof.
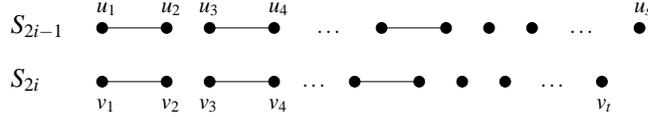

Figure 1: The ordering of the vertices of S_{2i-1} and S_{2i}.

For any $1 \leq i \leq k$, we represent G_i as the intersection graph of 2-dimensional boxes. We order the vertices $u_1, \ldots, u_s$ of S_{2i-1} and the vertices $v_1, \ldots, v_t$ of S_{2i} as depicted in Figure 1 (recall that S_{2i-1} and S_{2i} both induce a graph with maximum degree at most one). Let r be the maximum of s and t. For every j such that u_{2j-1} and u_{2j} are adjacent in S_{2i-1}, u_{2j-1} is represented by the box $\{-r+2j-1\} \times [-2j+2, -2j+1]$ and u_{2j} is represented by the box $[-r+2j-1, -r+2j] \times \{-2j+1\}$. If a vertex u_j is isolated in S_{2i-1}, it is represented by the point $(-r+j, -j+1)$.

Similarly, for every j such that v_{2j-1} and v_{2j} are adjacent in S_{2i}, v_{2j-1} is represented by the box $[2j-2, 2j-1] \times \{r-2j+1\}$ and v_{2j} is represented by the box $\{2j-1\} \times [r-2j, r-2j+1]$. If a vertex v_j is isolated in S_{2i}, it is represented by the point $(j-1, r-j)$ (see Figure 2(b) for an example).

Observe that :

(1) the boxes of two adjacent vertices u_{2j-1} and u_{2j} intersect in $(-r+2j-1, -2j+1)$;

(2) the boxes of two adjacent vertices v_{2j-1} and v_{2j} intersect in $(2j-1, r-2j+1)$;

(3) the boxes of all the other pairs of vertices colored with $2i-1$ or $2i$ are not intersecting.

(4) the top-right corner of the box of u_j is the point $(-r+j, -j+1)$ and the bottom-left corner of the box of v_j is the point $(j-1, r-j)$

We now have to represent the vertices from K_i. We represent the vertices having no neighbor outside K_i by the point $(0,0)$. If a vertex u from K_i has only one neighbor outside K_i, say $u_j \in S_{2i-1}$, we represent u by the box $[-r+j, 0] \times [-j+1, 0]$. If a vertex v from K_i has only one neighbor outside K_i, say $v_j \in S_{2i}$, we represent v by the box $[0, j-1] \times [0, r-j]$. If a vertex w of K_i has one neighbor $u_j \in S_{2i-1}$ and
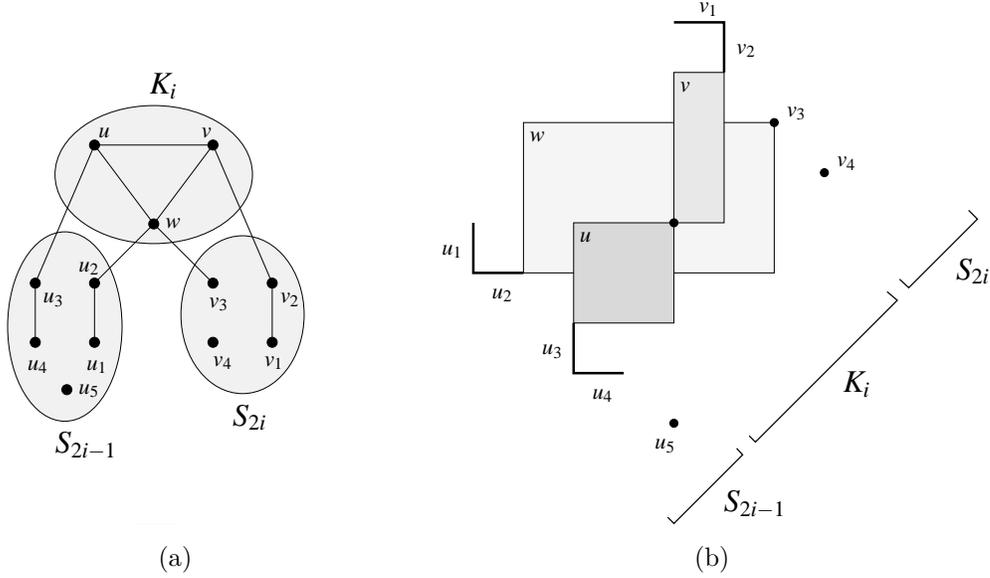
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Figure 2: (a) A graph G_i and (b) a representation of G_i as the intersection graph of 2-dimensional boxes.

one neighbor $v_\ell \in S_{2i}$, we represent w by the box $[-r+j, \ell-1] \times [-j+1, r-\ell]$ (see Figure 2(b) for an example).

The boxes representing the vertices from K_i are pairwise intersecting, since they all contain the point $(0,0)$. Moreover, using Observation (4) above, the box of every vertex v from K_i only intersects the boxes of the neighbors of v. Hence, G_i is the intersection graph corresponding to this representation, and so G_i has boxicity two, which concludes the proof.

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The best known lower bound for the boxicity of graphs with maximum degree Δ was given by Roberts [8]. Consider the graph H_{2n} obtained by removing a perfect matching from a clique of $2n$ vertices. If this graph has boxicity $k \le n-1$, let $G_1, \ldots, G_k$ be interval graphs such that $H_{2n} = G_1 \cap \ldots \cap G_k$. Since $k \le n-1$ and H_{2n} have n non-edges, two non-edges of H_{2n} have to lie in the same interval graph, say G_i. This is impossible since otherwise G_i contains an induced cycle of length four and is not an interval graph. Hence, $\mathrm{box}(H_{2n}) \ge n \ge \left\lceil \frac{1}{2}\Delta(H_{2n}) \right\rceil$.

Chandran et al. [2] conjectured that for any graph G, $\mathrm{box}(G) \le O(\Delta)$. It is interesting to remark that this conjecture is true when the graphs $G_1, \ldots, G_k$ with $G = \cap_{1 \le i \le k} G_i$ are only required to be chordal. McKee and Scheinerman [6] defined the *chordality* of a graph G, denoted by $\mathrm{chord}(G)$, as the smallest k such that G is the intersection of k chordal graphs. Since a graph is an interval graph if and

only if it is chordal and its complement is a comparability graph, we clearly have chord$(G) \leq$ box(G) for any graph G. McKee and Scheinerman proved that the chordality of a graph is bounded by its chromatic number. As a corollary, it is easy to show that for any graph G with maximum degree Δ, chord$(G) \leq \Delta$.

We conclude with general remarks. We denote by $a(G)$ the *arboricity* of G, that is the minimum number of induced forests into which the edges of G can be partitioned. For outerplanar graphs, planar graphs, graphs with bounded treewidth, and graphs with bounded degree, the boxicity seems to be bounded by the arboricity. Unfortunately it seems to be false in general: there exists trees with boxicity at least two, and graphs with arboricity two and boxicity at least three. This leads to two natural questions:

1. is there a constant $\kappa \geq 1$, such that any graph G satisfies box$(G) \leq a(G) + \kappa$?

2. is there a constant $\lambda > 1$, such that any graph G satisfies box$(G) \leq \lambda a(G)$?

A positive answer to the second question (and thus to the first), would imply that for any graph G with maximum degree Δ, box$(G) \leq \lambda \left\lceil \frac{\Delta+1}{2} \right\rceil$, proving the conjecture of Chandran et al. [2].

References

[1] L.S. Chandran and N. Sivadasan, *Boxicity and treewidth*, J. Combin. Theory Ser. B **97(5)** (2007) 733-744.

[2] L.S. Chandran, M.C. Francis, and N. Sivadasan, *Boxicity and maximum degree*, J. Combin. Theory Ser. B **98** (2008) 443-445.

[3] M.B. Cozzens and F.S. Roberts, *Computing the boxicity of a graph by covering its complement by cointerval graphs*, Discrete Applied Math. **6** (1983) 217-228.

[4] L.C. Freeman, *Spheres, cubes and boxes: graph dimensionality and network structure*, Social Networks **5** (1983), 139-156.

[5] J. Kratochvíl, *A special planar satisfiability problem and a consequence of its NP-completeness*, Discrete Appl. Math. **52** (1994) 233–252.

[6] T.A. McKee and E.R. Scheinerman, *On the Chordality of a Graph*, J. Graph Theory **17(2)** (1993) 221-232.

[7] R.J. Opsut and F.S. Roberts, *On the fleet maintenance, mobile radio frequency, task assignment, and traffic phasing problems*, in G. Chartrand, et al (eds.), The Theory and Applications of Graphs, Wiley, New York, 1981, 479-492.

[8] F.S. Roberts, *On the boxicity and cubicity of a graph*, In: Recent Progresses in Combinatorics, Academic Press, New York, 1969, 301–310.

[9] F.S. Roberts, *Discrete Mathematical Models, with Applications to Social, Biological, and Environmental Problems*, Prentice-Hall, Englewood Cliffs, New Jersey, 1976.

[10] E.R. Scheinerman, *Intersection classes and multiple intersection parameters*, PhD thesis, Princeton University, 1984.

[11] C. Thomassen, *Interval representations of planar graphs*, J. Combin. Theory Ser. B **40** (1986) 9–20.