Dynamic list coloring of bipartite graphs

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Abstract

A *dynamic coloring* of a graph is a proper coloring of its vertices such that every vertex of degree more than one has at least two neighbors with distinct colors. The least number of colors in a dynamic coloring of G, denoted by $\chi_2(G)$, is called the *dynamic chromatic number* of G. The least integer k, such that if every vertex of G is assigned a list of k colors, then G has a proper (resp. dynamic) coloring in which every vertex receives a color from its own list, is called the *choice number* of G, denoted $ch(G)$ (resp. the *dynamic choice number*, denoted $ch_2(G)$). It was recently conjectured [S. Akbari *et al.*, *On the list dynamic coloring of graphs*, Discrete Appl. Math. (2009)] that for any graph G, $ch_2(G) = \max(ch(G), \chi_2(G))$. In this short note we disprove this conjecture. We first give an example of a small planar bipartite graph G with $ch(G) = \chi_2(G) = 3$ and $ch_2(G) = 4$. Then, for any integer $k \geq 5$, we construct a bipartite graph G_k such that $ch(G_k) = \chi_2(G_k) = 3$ and $ch_2(G) \geq k$.

Keywords: Dynamic coloring, list coloring, incidence graphs.

For a graph G, the *incidence graph* of G, denoted $G^\star$, is the graph obtained from G by subdividing each of its edges exactly once (i.e. by replacing each edge by a path of length two). The new vertices are called *the middle vertices* of $G^\star$ and the other vertices are called *the original vertices*. We first remark that for any graph G, we have $\chi_2(G^\star) \geq \chi(G)$ and $ch_2(G^\star) \geq ch(G)$: since the middle vertices of $G^\star$ have degree two, their neighbors (the end-vertices of an edge of G) must receive distinct colors in any dynamic coloring of $G^\star$. Among other consequences, there exist bipartite graphs with arbitrarily large dynamic chromatic number (take the incidence graphs of complete graphs, for instance).

Consider a (possibly improper) coloring of the edges of G such that the set of edges incident to any vertex of degree more than one contains at least two distinct colors. We use $ch_2^*(G)$ to denote the smallest integer k such that if every edge of G is given a list of k colors, G has such a coloring with the additional property that every edge is assigned a color from its list. The following lemma relates $ch(G)$, $ch_2^*(G)$, and $ch_2(G^\star)$:

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Lemma 1. *For any graph G, $ch_2(G^\star) \leq \max(ch(G), ch_2^*(G)+2)$. In particular, $ch(G) \leq ch_2(G^\star) \leq \max(5, ch(G))$.*

Proof. Let L be an assignment of lists of colors to the vertices of $G^\star$ so that for any original vertex u, $|L(u)| \geq ch(G)$, and for any middle vertex v, $|L(v)| \geq ch_2^*(G)+2$. Let c be a proper coloring of G such that for any vertex $u \in G$, $c(u) \in L(u)$ (such a coloring exists by the definition of $ch(G)$). For any middle vertex w of $G^\star$ with neighbors u, v, set $L'(w) = L(w) \setminus \{c(u), c(v)\}$. By the definition of L and L', we have $|L'(u)| \geq ch_2^*(G)$ for any middle vertex u, so we can extend c to the middle vertices of $G^\star$, in such way that every original vertex u is adjacent to two middle vertices v and w with $c(v) \neq c(w)$. Hence, we obtain a dynamic coloring c of $G^\star$ such that for any vertex v, $c(v) \in L(v)$.

We now prove that for any graph G, $ch_2^*(G) \leq 3$. Given a list assignment of 3 colors to every edge of G, we greedily color the edges of G in the following way: for any non-colored edge uv, pick a color different from one of the colors appearing on the edges incident with u, and from one of the colors appearing on the edges incident with v (if such colors exist, otherwise pick an arbitrary color in the list of uv). Since the list of uv contains three colors, this is always possible, and the coloring obtained is such that vertices with degree more than one have at least two distinct colors among the edges they are incident with. □

Similarly, we can prove that $\chi(G) \leq \chi_2(G^\star) \leq \max(\chi(G), ch_2^*(G)+2) \leq \max(5, \chi(G))$. On the other hand $\chi_2(G)$ and $\chi_2(G^\star)$ are not necessarily close: if $G = K_n^\star$, then as remarked above $\chi_2(G) = n$, but $\chi_2(G^\star) \leq 4$. However we can prove the following lemma:

Lemma 2. *For any graph G with $2 \leq \chi_2(G) \leq 3$, we have $\chi_2(G^\star) = 3$.*

Proof. Since $\chi_2(G) \geq 2$, G contains at least one edge and so $\chi_2(G^\star) \geq 3$. Consider now a dynamic 3-coloring c of G, and define the dynamic 3-coloring $c^\star$ of $G^\star$ as follows: for any original vertex v of $G^\star$, set $c^\star(v) = c(v)$ and for any middle vertex w corresponding to an edge uv of G, set $c^\star(w) = i$, where $\{i\} = \{1,2,3\} \setminus \{c(u), c(v)\}$ (this is well-defined since c is a proper coloring of G). Such a coloring is proper and since middle vertices have two neighbors of distinct colors, we only have to check that this is also the case for every original vertex u of degree at least two. But since c is a dynamic coloring, u has two neighbors v and w in G with $c(v) \neq c(w)$. By the definition of $c^\star$, the middle vertices on uv and uw also have distinct colors, and they are both neighbors of u in $G^\star$. □

We now use Lemmas 1 and 2 to construct a small bipartite planar graph G (on 65 vertices), such that $ch(G) = \chi_2(G) = 3$ and $ch_2(G) = 4$. Consider the graph H_{ij} of Figure 1, and observe that if the vertices u_{ij} and v_{ij} are colored with colors $i > 1$ and $j > 1$, such that $i \neq j$, then this coloring does not extend to x_{ij} and y_{ij} if they are both given the list $1ij$. Take 9 copies of H_{ij} for $(i,j) \in \{2,3,4\} \times \{5,6,7\}$, and identify all the vertices u_{ij} into a single vertex u^*, and all the vertices v_{ij} into a single vertex v^*. This new graph is called H. Using the observation above, if u^* and v^* are given the lists 234 and 456 respectively, and if x_{ij} and y_{ij}, $(i,j) \in \{2,3,4\} \times \{5,6,7\}$, are given the list $1ij$, then H cannot be properly colored with these lists. As a consequence, H is not 3-choosable.

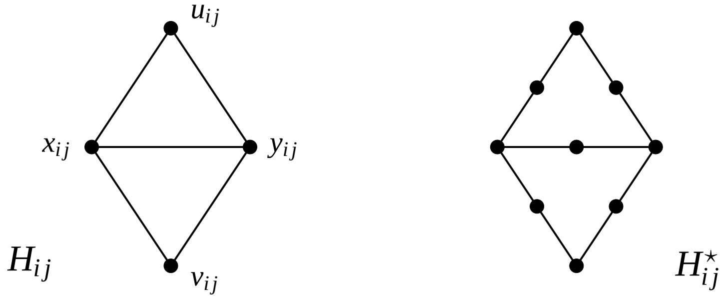

Figure 1: The graphs H_{ij} and $H^\star_{ij}$.

On the other hand it is easy to see that H is 4-choosable (since it is 3-degenerate), so $ch(H) = 4$.

Set $G = H^\star$. Since G is bipartite and planar, $ch(G) \leq 3$ by a result of Alon and Tarsi [2]. It is not difficult to check that G is not 2-choosable (using [3] for instance) so $ch(G) = 3$. We now prove that $\chi_2(G) = 3$. Using Lemma 2 it is enough to prove that $\chi_2(H) \leq 3$. But since every vertex of H is in a triangle, $\chi_2(H) = \chi(H)$, and since H is 3-colorable, we have $\chi_2(G) = 3$.

Since $ch(H) = 4$, we have $ch_2(G) \geq 4$ and in order to conclude, it is sufficient to prove that $ch_2(G) \leq 4$. By properly coloring the edges of each 4-cycle $u_{ij}x_{ij}v_{ij}y_{ij}$, $(i,j) \in \{2,3,4\} \times \{5,6,7\}$, and giving arbitrary colors to the edges $x_{ij}y_{ij}$, we see that $ch_2^*(H) = 2$. Hence, by Lemma 1, $ch_2(G) \leq 4$. As a consequence, G is a bipartite planar graph on 65 vertices such that $ch(G) = \chi_2(G) = 3$ and $ch_2(G) = 4$.

Gutner [4] constructed a 3-colorable but not 4-choosable planar graph H' on 75 vertices and 219 edges. Its incidence graph $G' = H'^\star$ is a bipartite planar graph on 294 vertices and using the same ideas as above, we can show that $\chi_2(G') = ch(G') = 3$ and $ch_2(G') = 5$.

Let $k \geq 5$ be an integer. We will now generalize the ideas of the previous paragraphs to construct a bipartite graph G_k such that $\chi_2(G_k) = ch(G_k) = 3$ and $ch_2(G_k) \geq k$. Let $K^-_{\ell,\ell}$ be the graph obtained from the complete bipartite graph $K_{\ell,\ell}$ by removing an edge. Observe that for any $\ell \geq 1$, we have $ch(K^-_{\ell+1,\ell+1}) \leq ch(K^-_{\ell,\ell}) + 1$. To see this, assume that each vertex of $K^-_{\ell+1,\ell+1}$ has a list of at least $ch(K^-_{\ell,\ell}) + 1 \geq 2$ colors. Choose two adjacent vertices u, v of degree $\ell + 1$ in $K^-_{\ell+1,\ell+1}$, and color them with distinct colors from their lists. Now, remove the color of u (resp. v) from the lists of the neighbors of u (resp. v), and complete the coloring by observing that the graph obtained from $K^-_{\ell+1,\ell+1}$ by removing u and v is precisely $K^-_{\ell,\ell}$ and that each vertex has a list of size at least $ch(K^-_{\ell,\ell})$.

Erdős *et al.* [3] constructed complete bipartite graphs with arbitrarily large choice number. Combining this result with the previous remark, we obtain that the function $t : k \mapsto \min\{\ell \mid ch(K^-_{\ell,\ell}) = k\}$ is well-defined for integers $k \geq 2$. Consider the bipartite graph H_k depicted in Figure 2. It is obtained from the complete bipartite graph $K_{\ell,\ell}$,

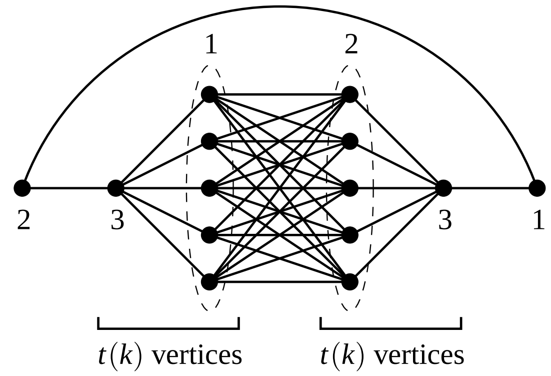

Figure 2: The graph H_k.

$\ell = t(k)$, by removing an edge between two vertices and adding a path of length three between them. The coloring described in Figure 2 shows that $\chi_2(H_k) \le 3$. Using Lemma 2, the graph $G_k = H_k^\star$ satisfies $\chi_2(G_k) = 3$. The graph G_k is easily seen to be 3-choosable (assign to its original vertices arbitrary colors from their lists, and assign to the middle vertices colors distinct from their two neighbors) but not 2-choosable (see [3]), which implies $\chi_2(G_k) = ch(G_k) = 3$.

We now prove that $ch_2(G_k) = k$. By definition, $ch(K^-_{\ell,\ell}) = k$, and since colorings of $K^-_{\ell,\ell}$ can be extended to H_k, we also have $ch(H_k) = k$. Hence, $ch_2(G_k) \ge k$. On the other hand, Lemma 1 implies that $ch_2(G_k) \le k$. As a consequence, G_k is a bipartite graph such that $\chi_2(G_k) = ch(G_k) = 3$ and $ch_2(G_k) = k$.

However, since the function t is not precisely known, we can only prove the existence of such a graph. Nevertheless, we can construct a bipartite graph G_k such that $\chi_2(G_k) = ch(G_k) = 3$ and $ch_2(G_k) \ge k$ by taking $\ell = \binom{2k-3}{k-1} + 1$. It can be proved [3] that in this case $ch(K^-_{\ell,\ell}) \ge ch(K_{\ell-1,\ell-1}) \ge k$ and the result follows.

We conclude with a remark on the parameter ch_2^* introduced before Lemma 1. In the proof of this lemma, it is shown that for any graph G, we have $ch_2^*(G) \le 3$. On the other hand, the construction of the bipartite planar graph G proved the existence of graphs with $ch_2^*(G) = 2$. A larger class of graphs with this property includes the graphs obtained from a bipartite Eulerian graph by repeatedly adding vertices of degree at least two (and all the supergraphs of graphs constructed this way, since adding edges to a graph G with minimum degree two and $ch_2^*(G) = 2$ leaves ch_2^* unchanged). A natural question is whether the recognition of graphs G with $ch_2^*(G) = 2$ is polynomial or not.

Additional remarks I would like to thank Frédéric Maffray for interesting discussions about the question above, after this paper was originally submitted. Let $\chi_2^*(G)$ be the least number of colors in a (possibly improper) coloring of the edges of G such that no vertex of degree at least two has all the edges incident to it colored the same. We clearly have

$\chi_2^*(G) \leq ch_2^*(G)$. As a partial answer to the previous question, we proved that $\chi_2^*(G) \leq 2$ precisely if G has no component isomorphic to an odd cycle. The idea of the proof is to either color a perfect matching with color 1 and the remaining edges with color 2, or (if the graph has no perfect matching) to use Tutte's theorem together with an appropriate edge-coloring. However, the proof does not seem to extend to a precise characterization of the graphs G satisfying $ch_2^*(G) = 2$.

References

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