

Game colouring of the square of graphs

Louis Esperet* and Xuding Zhu†

* *Institute for Theoretical Computer Science, Charles University, Prague, Czech Republic*

† *Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan*

Feb. 2009

Abstract

This paper studies the game chromatic number and game colouring number of the square of graphs. In particular, we prove that if G is a forest of maximum degree $\Delta \geq 9$, then $\chi_g(G^2) \leq \text{col}_g(G^2) \leq \Delta + 3$, and there are forests G with $\text{col}_g(G^2) = \Delta + 3$. It is also proved that for an outerplanar graph G of maximum degree Δ , $\chi_g(G^2) \leq \text{col}_g(G^2) \leq 2\Delta + 14$, and for a planar graph G of maximum degree Δ , $\chi_g(G^2) \leq \text{col}_g(G^2) \leq 23\Delta + 75$.

1 Introduction

The game chromatic number of a graph is defined through a two-player game: let G be a graph and C be a set of colours. Alice and Bob take turns colouring uncoloured vertices of G , with Alice having the first move. Each move colours one uncoloured vertex, subject to the condition that two adjacent vertices cannot be coloured with the same colour. Alice wins the game if eventually every vertex is coloured. Bob wins the game if some uncoloured vertex x cannot be coloured anymore (each colour in C has been assigned to some neighbour of x). The *game chromatic number* $\chi_g(G)$ of G is the minimum k for which Alice has a winning strategy with a set of k colours in this game.

The game chromatic number has been widely studied over the last decade. Upper and lower bounds for the game chromatic number of many classes of graphs have been obtained. For a class \mathcal{K} of graphs, let

$$\chi_g(\mathcal{K}) = \max\{\chi_g(G) : G \in \mathcal{K}\}.$$

We denote by \mathcal{F} the family of forests, by \mathcal{P} the family of planar graphs, by \mathcal{Q} the family of outerplanar graphs. It is known that $\chi_g(\mathcal{F}) = 4$ [6], $6 \leq \chi_g(\mathcal{Q}) \leq 7$ [7, 9], $8 \leq \chi_g(\mathcal{P}) \leq 17$ [9, 13]. For two graphs G, G' , the *Cartesian product* $G \square G'$ has vertex set $\{(x, x') : x \in V(G), x' \in V(G')\}$ and (x, x') is adjacent to (y, y') if either $x = y$ and $x'y' \in E(G')$ or $xy \in E(G)$ and $x' = y'$. The game chromatic number of the Cartesian product of graphs was studied in [1, 14]. For two classes $\mathcal{K}, \mathcal{K}'$ of graphs, $\mathcal{K} \square \mathcal{K}' = \{G \square G' : G \in \mathcal{K}, G' \in \mathcal{K}'\}$. It was proved in [14] that $\chi_g(\mathcal{F} \square \mathcal{F}) \leq 10$ and $\chi_g(\mathcal{P} \square \mathcal{P}) \leq 105$.

email: esperet@kam.mff.cuni.cz, zhu@math.nsysu.edu.tw

In this paper, we are interested in the game chromatic number of the square of graphs. Suppose $G = (V, E)$ is a graph. The *square* of G , denoted by G^2 , is a graph with vertex set V in which two distinct vertices x, y are adjacent if $d_G(x, y) \leq 2$, i.e., either $xy \in E$ or x, y have common neighbour. Hence, $\chi_g(G^2)$ can be equivalently defined as the minimum number of colours such that Alice has a winning strategy in a variation of the game where the requirement is that at any step, vertices at distance at most two in G cannot be coloured by the same colour. Section 2 gives an upper bound on $\chi_g(G^2)$ in terms of $\Delta(G)$ and the game colouring number of G . Section 3 discusses the game chromatic number of the square of forests. Section 4 considers the square of outerplanar graphs. Section 5 studies pseudo-partial 2-trees.

2 Game colouring number

The game colouring number of a graph is a variation of the game chromatic number, first formally introduced in [11] as a tool in the study of game chromatic number. It is also defined through a two-player game: Alice and Bob take turns marking the vertices of G . Each move marks one unmarked vertex. The game ends if all vertices of G are marked. The *game colouring number* of G , $\text{col}_g(G)$ is the least integer k such that Alice has a strategy for the marking game so that at any moment, any unmarked vertex has at most $k - 1$ marked neighbours. It is obvious that for any graph G , $\chi_g(G) \leq \text{col}_g(G)$. For a class \mathcal{K} of graphs, let $\text{col}_g(\mathcal{K}) = \max\{\text{col}_g(G) : G \in \mathcal{K}\}$. For many classes \mathcal{K} of graphs, the best known upper bound for $\chi_g(\mathcal{K})$ are obtained by considering $\text{col}_g(\mathcal{K})$. In this paper, we shall also obtain upper bounds for the game chromatic number of squares of graphs by studying their game colouring number. Observe that $\text{col}_g(G^2)$ can be equivalently defined as the least integer k such that Alice has a strategy for the marking game so that at any moment, any unmarked vertex has at most $k - 1$ marked vertices at distance at most 2 in G .

Theorem 2.1 *If G has game colouring number k and maximum degree Δ , then $\chi_g(G^2) \leq \text{col}_g(G^2) \leq (k - 1)(2\Delta - k + 1) + 1$.*

Proof Assume that Alice has a strategy for the marking game on G to ensure that at any moment of the game, any unmarked vertex has at most $k - 1$ marked neighbours in G . We shall show that by using the same strategy, Alice can ensure that at any moment of the game, any unmarked vertex has at most $(k - 1)(2\Delta - k + 1)$ marked vertices at distance at most 2 in G . Indeed, if v is an unmarked vertex, then let $N_M(v)$ be the set of marked neighbours of v in G , and $N_U(v)$ be the set of unmarked neighbours of v in G . Each vertex of $N_M(v)$ has at most $\Delta - 1$ marked neighbours, and each vertex of $N_U(v)$ has at most $k - 1$ marked neighbours. Hence, v has at most $|N_M(v)|(\Delta - 1) + |N_M(v)| + (k - 1)|N_U(v)| \leq \Delta(k - 1) + |N_M(v)|(\Delta - k + 1)$ marked vertices at distance at most two. As $|N_M(v)| \leq k - 1$, there are at most $(k - 1)(2\Delta - k + 1)$ such vertices. \square

3 Game colouring of the square of forests

For special classes of graphs, the upper bound for $\chi_g(G^2)$ in Theorem 2.1 can usually be improved. This section proves a better upper bound for $\chi_g(G^2)$ when G is a forest.

Theorem 3.1 *If G is a forest with maximum degree $\Delta \geq 9$, then $\Delta + 1 \leq \chi_g(G^2) \leq \text{col}_g(G^2) \leq \Delta + 3$.*

For any forest G , $\omega(G^2) = \Delta + 1$. Therefore $\chi_g(G^2) \geq \Delta + 1$. Assume $G = (V, E)$ is a forest with $\Delta \geq 9$. To prove that $\text{col}_g(G^2) \leq \Delta + 3$, we shall give a strategy for Alice for the marking game on G^2 , so that at any moment of the game, each unmarked vertex has at most $\Delta + 2$ marked neighbours in G^2 .

If G is not a tree, then we may add some edges to G to obtain a tree. Thus we may assume that G is a tree. Alice's strategy is a variation of the activation strategy, which is widely used in the study of colouring games and marking games. She keeps track of a set $V_a \subseteq V$ of active vertices, which always induces a subtree of G . When a vertex v is added to V_a , we say that v is *activated*. Vertices in V_a are called *active* vertices, and other vertices are called *inactive*.

Choose a vertex r of G as the *root*, and view G as a rooted tree. For a vertex x , $f^1(x)$ (abbreviated as $f(x)$) is the father of x and for $i \geq 2$, let $f^i(x) = f(f^{i-1}(x))$. For convenience, we let $f(r) = r$. The vertices in $\{f^i(x) : i \geq 1\}$ are called the *ancestors* of x . Let $S(x)$ be the set of sons of x , and let $S^2(x) = \cup_{y \in S(x)} S(y)$ be the set of grandsons of x .

Alice's strategy:

- Initially she sets $V_a = \{r\}$, and marks r .
- Assume Bob has just marked a vertex x and there are still unmarked vertices. Let P_x be the unique path from x to the nearest ancestor y of x that is in V_a . In particular, if $x \in V_a$, then $x = y$ and P_x consists of the single vertex x . Alice adds all the vertices of P_x to V_a , and marks the first unmarked vertex from the sequence: $f^2(y), f(y), y, z^*, v$, where v is an unmarked vertex with no unmarked ancestors, and z^* is defined as follows: Let $Z = \{z \in S(y) : z \text{ is unmarked and } |(S(z) \cup S^2(z)) \cap V_a| \text{ is maximum among all unmarked sons of } y\}$. Let M be the set of marked vertices. Then z^* is a vertex in Z for which $|(S(z^*) \cup S^2(z^*)) \cap M|$ is maximum. In case $Z = \emptyset$, then ignore the vertex z^* in the sequence.

This completes the description of Alice's strategy. In the following, we shall show that by using this strategy, each unmarked vertex has at most $\Delta + 2$ marked neighbours in G^2 (or equivalently, each unmarked vertex has at most $\Delta + 2$ marked vertices at distance one or two in G).

For each vertex x marked by Bob, there is a path P_x defined as above. We say that a vertex w *made a contribution* to $f(w)$ and $f(w)$ *received a contribution* from w , if one of the following holds:

1. $(w, f(w))$ is an edge in P_x for some x .
2. $w = x'$ is the last vertex of P_x for some x and Alice marked $f(x')$ or $f^2(x')$ in that step.
3. $w = f(x')$ is the father of the last vertex x' of P_x for some x and Alice marked $f^2(x') = f(w)$ in that step.

Lemma 3.2 *Assume Alice has just finished a move and y has two active sons. Then $f^2(y)$ is marked.*

Proof When the first son of y is activated, then y and all its ancestors are activated. When the second son of y is activated, then the corresponding path P_x ends at y , and by the strategy, Alice marks $f^2(y)$, provided that $f^2(y)$ was not marked earlier. \square

Lemma 3.3 *Assume Alice has just finished a move, and one of $y, f(y)$ is an unmarked vertex. Then the following holds:*

- (1) y has at most 3 active sons.
- (2) $S(y) \cup S^2(y)$ contains at most 6 active vertices. Moreover, if $S(y) \cup S^2(y)$ does contain 6 active vertices, then y has 3 active sons, each of which has one active son.

Proof The first contribution to y ensures that $y, f(y)$ and $f^2(y)$ are all active, and each further contribution marks at least one of these three vertices (as long as y is unmarked). Since y or $f(y)$ is unmarked, y received at most three contributions. During each of the three corresponding moves of Alice, at most one vertex of $S(y)$ and at most one vertex of $S^2(y)$ are activated. So $S(y)$ contains at most three active vertices and $S^2(y)$ contains at most three active vertices. In case $S(y) \cup S^2(y)$ does contain 6 active vertices, then y has three active sons, each of which has one active son. \square

Lemma 3.4 *Assume Alice has just finished a move, and one of $y, f(y)$ is an unmarked vertex. Then y has at most one unmarked son x such that $S(x) \cup S^2(x)$ contains more than 2 active vertices.*

Proof Assume to the contrary that y and $f(y)$ are not both marked and y has two unmarked sons x_1, x_2 such that for each $j = 1, 2$, $S(x_j) \cup S^2(x_j)$ contains more than 2 active vertices. For $j = 1, 2$, if a vertex in $S(x_j) \cup S^2(x_j)$ is activated, the corresponding path P_x ends at x_j or a vertex $z \in S(x_j)$. Hence x_j receives a contribution. Since x_j is unmarked, x_j passes the contribution to y . As $S(x_j) \cup S^2(x_j)$ contains more than 2 active vertices, there are at least two steps in which some vertex in $S(x_j) \cup S^2(x_j)$ is activated. Hence y received at least 4 contributions. As remarked in the proof of Lemma 3.3, if y received 4 contributions, then both $y, f(y)$ are marked. \square

Lemma 3.5 *Assume Alice has just finished a move. Then the following holds:*

- y has at most two unmarked sons x for which $S(x) \cup S^2(x)$ contains more than 2 active vertices.
- If y has 3 active sons, then y has at most one unmarked son x for which $S(x) \cup S^2(x)$ contains more than 2 active vertices. If y has 4 or more active sons, then for each unmarked $x \in S(y)$, $S(x) \cup S^2(x)$ contains at most two active vertices and contains at most one marked vertex.

Proof By Lemma 3.4, before y and $f(y)$ are both marked, y has at most one unmarked son x such that $S(x) \cup S^2(x)$ contains more than 2 active vertices. Therefore at the moment the last of the two vertices y and $f(y)$ is marked, y has at most two unmarked sons x for which $S(x) \cup S^2(x)$ has more than 2 active vertices. Moreover, if y does have two unmarked sons x for which $S(x) \cup S^2(x)$ contains more than 2 active vertices, then y has only two active unmarked sons.

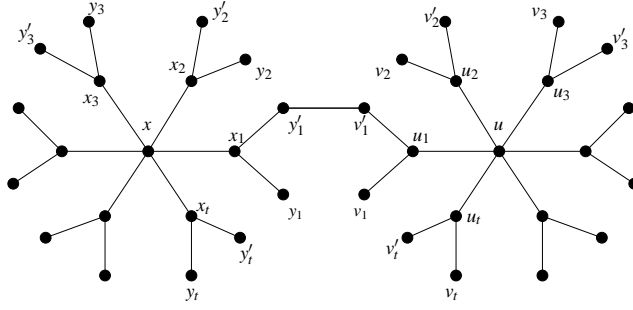


Figure 1: A tree T with $\text{col}_g(T^2) = \Delta + 3$.

Assume that at the moment that the last of the two vertices y and $f(y)$ is marked, y has two unmarked sons, say x_1 and x_2 , such that $S(x_i) \cup S^2(x_i)$ contains more than 2 active vertices ($i = 1, 2$). By Lemma 3.2, $f^2(y)$ is marked.

Suppose the third son x_3 of y is activated. Since $f^2(y), f(y), y$ are all marked, by the strategy, one of x_1 and x_2 , say x_1 , will be marked. At the time x_3 is activated, $S(x_3) \cup S^2(x_3)$ contains at most two active vertices and at most one marked vertex. If one more vertex of $S(x_3) \cup S^2(x_3)$ is activated or marked, then Alice should have marked x_3 . When the fourth son x_4 of y is activated, Alice should have marked x_2 . Once both x_1 and x_2 are marked, then for any son x of y , if $S(x) \cup S^2(x)$ contains more than 2 active vertices or contains more than one marked vertex, Alice should have marked x . \square

Lemma 3.6 *Assume $\Delta(G) \geq 9$. If Alice has just finished a move and x is an unmarked vertex, then there are at most $\Delta + 1$ marked vertices at distance at most 2 (in G) from x .*

Proof By Lemma 3.3, $S(x) \cup S^2(x)$ contains at most 6 active vertices, and so at most 6 marked vertices since after any of Alice's moves all the marked vertices are active. The other marked vertices at distance at most 2 from x are $f(x)$ and the neighbours of $f(x)$. By Lemma 3.5, if $S(x) \cup S^2(x)$ contains at least 2 two marked vertices then $f(x)$ has at most 3 active sons (including x), hence the set $N[f(x)] - \{x\}$ contains at most 4 marked vertices : $f(x)$, $f^2(x)$, and two sons of $f(x)$. So in this case there are at most $4 + 6 = 10 \leq \Delta + 1$ marked vertices at distance at most 2 from x . If $S(x) \cup S^2(x)$ contains at most one marked vertex, then again there are at most $\Delta + 1$ marked vertices at distance at most 2 from x . \square

After Bob's move, an unmarked vertex x has at most $\Delta + 2$ active vertices that are of distance at most 2 from x . This proves that the game colouring number of the square of a forest F is at most $\Delta + 3$.

The bound $\text{col}_g(G^2) \leq \Delta + 3$ is tight for trees. To see this, consider the graph depicted in Figure 1. By symmetry, we can assume that Alice does not mark x or x_i during her first move. Let $X = \{x_i, 1 \leq i \leq t\}$, $Y_i = \{y_i, y'_i\}$, and $Y = \bigcup_{1 \leq i \leq t} Y_i$. We say that Y_i has been marked if any of y_i and y'_i has been marked. Bob's strategy is the following : if there is an unmarked vertex x_i , such that Y_i is not marked, Bob marks y_i . Otherwise he just marks any u_j, v_j , or v'_j .

We now prove that if Bob follows this strategy, some unmarked vertex will be adjacent to at least $\Delta + 2$ marked vertices in T^2 at some point of the game.

After Bob's first move, the number of marked Y_i 's is one more than the number of marked x_i 's. If Alice marks an x_i whenever Bob marks Y_i , then eventually x will have too many marked neighbours in T^2 . So before all the x_i 's are marked, Alice needs to mark x at a certain move. Then before all the x_i 's are marked, if Bob has just finished a move, the number of marked Y_i 's is at least two more than the number of marked x_i 's.

Let x_i and x_j be the last vertices of X to be marked. Before x_i, x_j are marked, Bob has already marked y_i and y_j . Without loss of generality, assume that Alice chooses to mark x_i first, then Bob marks y_j and after his move, x_j is unmarked and has at least $\Delta + 2$ neighbours in T^2 .

4 Outerplanar graphs

A graph G is an outerplanar graph if G can be embedded in the plane in such a way that all the vertices of G lie on the boundary of the infinite face. This section gives an upper bound for $\chi_g(G^2)$ for outerplanar graphs.

Theorem 4.1 *Let G be an outerplanar graph with maximum degree Δ , then $\chi_g(G^2) \leq \text{col}_g(G^2) \leq 2\Delta + 14$.*

Let $G = (V, E)$ be an outerplanar graph with maximum degree Δ , and let $H = (V, E')$ be a maximal outerplanar graph containing G . Since H is a 2-tree, there exists an orientation \vec{H} of H such that:

- every vertex of \vec{H} has out-degree at most two;
- the two out-neighbours of any vertex, if they exist, are adjacent.

If a vertex x of H has two out-neighbours y, z , and \vec{yz} is an arc of H , then we say that z is the *major parent* of x , x is a *major son* of z , y is the *minor parent* of x , and x is a *minor son* of z . If x has only one out-neighbour z , then z is the major parent of x and x is a major son of z . For a vertex x , we denote by $f(x)$ (resp. $l(x)$) its major (resp. minor) parent, if it exists. We also define $S(x)$ as the set of in-neighbours of x and $S^2(x)$ as the set of in-neighbours of the vertices of $S(x)$.

Observation 4.2 *For every vertex $x \in \vec{H}$, at most two in-neighbours of x are minor sons of x . The minor sons of x , if any, are major sons of $f(x)$ or $l(x)$.*

This observation is an easy consequence of the definition of \vec{H} (see Figure 2, where only x_1 and x_t may be minor sons of x).

Let \vec{T} be the directed tree defined by the arcs $\{\overrightarrow{xf(x)}, x \in \vec{H}\}$. As in the previous section, Alice's strategy is a variation of the activation strategy and she will keep track of a set V_a of active vertices.

Alice's strategy

- At her first move, Alice will mark the root r of \vec{T} , and set $V_a = \{r\}$.

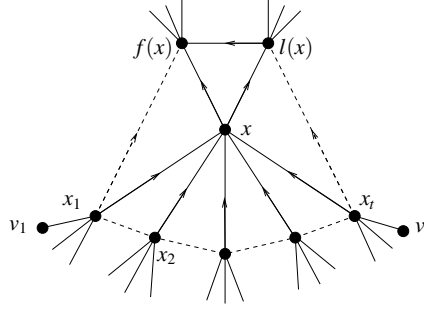


Figure 2: The neighbourhood of a vertex x in \vec{H} . The dashed arcs may not be there in the graph.

- Assume Bob just marked a vertex x . Let P_x be the path constructed as follows: At the beginning $P_x = \{x\}$. Let z be the last vertex of P_x . If z is inactive, then add $f(z)$ to P_x . Otherwise if $l(z)$ is inactive, add $l(z)$ to P_x . Eventually the procedure will stop and the last vertex y of P_x , as well as its parents, are all active (note that if z is active then $f(z)$ must be active). Alice adds all the vertices of P_x to V_a and marks the first unmarked vertex from the sequence $f(y), l(y), y, v$, where v is an unmarked vertex with no unmarked ancestors.

Lemma 4.3 *Let x be an unmarked vertex after a move of Alice, then there are at most $2\Delta + 12$ active vertices at distance one or two from x in G .*

Proof Assume to the contrary that after a move of Alice, there are at least $2\Delta + 13$ active vertices at distance one or two from an unmarked vertex x in G . Let $f(x), l(x)$ be the major and minor parents of x , and x_1, \dots, x_t are the sons of x (see Figure 2), where x_1 and x_t are minor sons of x . Let v_1 be the minor son of x_1 that is a major son of $f(x)$, and v_t be the minor son of x_t that is a major son of $l(x)$. (some of the vertices $f(x), l(x), x_1, x_t, v_1, v_t$ may not exist. In that case ignore them in the following discussion.)

Among these $2\Delta + 13$ vertices, 2Δ of them may be $f(x), l(x)$ and their neighbours (other than x). The other 13 are contained in $S(x) \cup S^2(x)$. Hence $S(x) \cup S^2(x) - \{x_1, x_t, v_1, v_t\}$ contains at least 9 active vertices. In each of Alice's move, at most two vertices in $S(x) \cup S^2(x)$ are activated. So there are at least 5 moves of Alice, in which some vertices in $S(x) \cup S^2(x) - \{x_1, x_t, v_1, v_t\}$ are activated.

In the first of such a move, if the activated vertex of $S(x) \cup S^2(x) - \{x_1, x_t, v_1, v_t\}$ is not a major son of x_t , then $f(x)$ is activated (unless $f(x)$ is activated before this move). If the activated vertex of $S(x) \cup S^2(x) - \{x_1, x_t, v_1, v_t\}$ is a major son of x_t , then $l(x)$ is activated (unless $l(x)$ is activated before this move).

After the second of such a move, $f(x)$ and $l(x)$ are both activated (by the previous paragraph, at least one of these two vertices is activated before this move).

After the third and the fourth of such moves, $f(x)$ and $l(x)$ are marked and x is activated (the latest time for x to be activated is in the third of such a move). In the fifth of these moves, x will be marked, in contrary to our assumption. \square

After Bob's move, an unmarked vertex has at most $2\Delta + 13$ active vertices at distance one or two in G . This proves that the game colouring number of the square of an outerplanar

graph with maximum degree Δ is at most $2\Delta + 14$.

Note that in the description and analyse of the strategy, we always use the graph H , which is a triangulated outerplanar graph obtained from G by adding some edges. But the degree of a vertex x refers to its degree in G , and Δ is the maximum degree of G .

5 Pseudo-partial 2-trees

The class of pseudo-partial k -trees is a class of graphs introduced in [12], as a generalization of partial k -trees. A graph $G = (V, E)$ is a *chordal graph* if there is a linear order, say v_1, v_2, \dots, v_n , on the vertex set V , such that for each i , the set $\{v_j : j < i, v_j v_i \in E\}$ induces a complete subgraph of G . By orienting the edges of G in such a way that an edge $v_i v_j$ is directed from v_i to v_j if and only if $i > j$, we obtain an oriented graph $\vec{G} = (V, \vec{E})$ which is acyclic and for each vertex v_i , its out-neighbours induce a transitive tournament. The converse is also true, i.e., a graph $G = (V, E)$ is a chordal graph if and only if G has an orientation $\vec{G} = (V, \vec{E})$ which is acyclic and the out-neighbours of each vertex induce a transitive tournament. For an oriented graph \vec{G} and a vertex u of \vec{G} , we denote the neighbours of u by $N_{\vec{G}}(u)$, the out-neighbours of u by $N_{\vec{G}}^+(u)$, and the in-neighbours of u by $N_{\vec{G}}^-(u)$. We denote the degree, out-degree and in-degree of u by $d_{\vec{G}}(u)$, $d_{\vec{G}}^+(u)$ and $d_{\vec{G}}^-(u)$, respectively. When the oriented graph \vec{G} is clear from the context we will drop the subscript.

Suppose a, b are integers such that $0 \leq a \leq b$. A connected graph $G = (V, E)$ is called an (a, b) -*pseudo-chordal graph* if there are two oriented graphs $\vec{G}_1 = (V, \vec{E}_1)$ and $\vec{G}_2 = (V, \vec{E}_2)$ on the same vertex set V such that the following is true:

- $E_1 \cap E_2 = \emptyset$ and $E = E_1 \cup E_2$, where E_i is the set of edges obtained from \vec{E}_i by omitting the orientations.
- \vec{G}_1 is acyclic and has a single sink r .
- \vec{G}_2 has maximum out-degree at most a , and maximum degree at most b .
- Let $N^+(x) = N_{\vec{G}_1}^+(x)$ be the set of out-neighbours of x in \vec{G}_1 , and let $\vec{G}^* = (V, \vec{E}_1 \cup \vec{E}_2)$. Then $N^+(x)$ induces a transitive tournament in \vec{G}^* .

A graph G is called an (a, b) -*pseudo-partial k -tree* if it is a subgraph of an (a, b) -pseudo-chordal graph in which the directed graph \vec{G}_1 in the definition has maximum out-degree at most k .

Note that any induced subgraph of an (a, b) -pseudo-chordal graph is still an (a, b) -pseudo-chordal graph. Therefore, an (a, b) -pseudo-partial k -tree can be equivalently defined as a spanning subgraph of an (a, b) -pseudo-chordal graph in which the directed graph \vec{G}_1 in the definition has maximum out-degree at most k .

It follows from the definition that if $b = 0$ (hence $a = 0$), then a $(0, 0)$ -pseudo-chordal graph is simply a chordal graph, and a $(0, 0)$ -pseudo-partial k -tree is simply a partial k -tree. However, for some $0 < a \leq b$, there are (a, b) -pseudo- k -trees which have arbitrarily large treewidth. For example, a result proved in [11] is equivalent to the statement that every planar graph is a $(3, 8)$ -pseudo-partial 2-tree. Nevertheless, for fixed a, b , the class of (a, b) -pseudo-chordal graphs does have some similarities with the class of chordal graphs, and the class of (a, b) -pseudo-partial k -trees does have similarities with the class of partial k -trees. We

shall explore such similarities, and use them to derive upper bounds for the game colouring number of the square of pseudo-partial 2-trees, and hence for the game colouring number of the square of planar graphs and partial 2-trees.

In this section, we apply some modifications on the activation procedure described in Section 3 of [12], to prove the following theorem:

Theorem 5.1 *Let G be an (a, b) -pseudo-partial 2-tree with maximum degree Δ , then $\text{col}_g(G^2) \leq (2b + a + 4)\Delta + 5a - b + 68$.*

Let $G = (V, E)$ be an (a, b) -pseudo-partial 2-tree, and let $\vec{G}_1 = (V, \vec{E}_1)$ and $\vec{G}_2 = (V, \vec{E}_2)$ be the two oriented graphs obtained from the pseudo-chordal supergraph of G as described in the decomposition of pseudo-partial k -trees. Recall that \vec{G}_2 has out-degree at most a and degree at most b , and that every vertex v has at most two out-neighbours in \vec{G}_1 . If x has two out-neighbours in \vec{G}_1 , then they are denoted by $f(x)$ and $l(x)$, and $\overrightarrow{l(v)f(v)}$ is an arc of \vec{E}_1 or \vec{E}_2 . We call $f(v)$ (resp. $l(v)$) a major (resp. minor) parent of v and v a major (resp. minor) son of $f(v)$ (resp. $l(v)$). If v has only one out-neighbour in \vec{G}_1 , then it is denoted by $f(x)$.

We now describe a strategy for Alice in the marking game on G^2 that ensures the score of the game is at most $(2b + a + 4)\Delta + 5a - b + 68$.

Two vertices v, v' are called *siblings* if they have the same parents, i.e., $f(v) = f(v')$ and $l(v) = l(v')$. For each vertex v , let $B(v)$ be the set of siblings of v .

Observation 5.2 (Lemma 1 of [12])

For any vertex x , its minor sons partition into at most $a + 2$ groups of siblings.

Alice's strategy is again a variation of the activation strategy. Besides the set of active vertices, Alice will also keep record of a function $t(v)$, which counts the number of contributions made by the set $B(v)$ to their parents. So the value of $t(v)$ will change in the process of the game. A vertex v is called *dummy* if $v, f(v), l(v)$ are all marked.

Alice's strategy

- Initially, Alice marks the sink r of \vec{G}_1 , sets $V_a = \{r\}$ and sets $t(v) = 0$ for all v .
- Assume Bob just marked a vertex x . Alice will create a directed path P_x using the following procedure. At the beginning $P_x = \{x\}$. Let z be the last vertex of P_x . If $z, f(z), l(z)$ are all active, or $f(z), l(z)$ are both dummy vertices, then the construction of P_x is complete. If at least one of $z, f(z), l(z)$ is inactive and none of $f(z), l(z)$ is a dummy vertex, then extend P_x by adding $f(z)$ or $l(z)$ to its end, depending on whether $t(z)$ is even or odd. For each vertex $v \in B(z)$, increase $t(v)$ by 1. If at least one of $z, f(z), l(z)$ is inactive and exactly one of $f(z), l(z)$ is not a dummy vertex, add that vertex to the end of P_x , and for each $v \in B(z)$, increase the value of $t(v)$ by 1. After the construction of P_x is completed, add all the vertices of P_x to V_a . Let y be the last vertex of P_x . Let v be a minimal unmarked vertex (that is, for every directed path in \vec{G}_1 starting at v , all the vertices of the path except v are marked). Alice marks the first unmarked vertex from the sequence $f(y), l(y), y, v$. If the marked vertex is $f(y)$ or $l(y)$, then for each vertex $u \in B(y)$, increase $t(u)$ by 1.

Similarly as before, if (w, w') is a directed edge in P_x for some P_x , then we say that w made a contribution to w' and w' received a contribution from w . Let x' be the last vertex of

P_x . If $f(x')$ ($l(x')$, resp.) is marked at this move, then we also say that $f(x')$ ($l(x')$, resp.) received a contribution from x' .

By this definition, it is easy to see that $t(v)$ counts the number of contributions made by vertices in $B(v)$ to their (common) parents.

The following observations follow from the strategy.

Observation 5.3 *A vertex x makes contributions to $f(x)$ and $l(x)$ only, and x can make at most two contributions to each of $f(x), l(x)$.*

Indeed, if x makes the first contribution to $f(x)$ (without loss of generality), then $f(x)$ becomes active. Before x makes a second contribution to $f(x)$, $l(x)$ has necessarily received a contribution from a vertex in $B(x)$. Hence x , $f(x)$ and $l(x)$ are all active and the relevant path P_z must have ended at x . Hence $f(x)$ becomes marked when it receives the second contribution from x .

Observation 5.4 *If a vertex x receives a contribution, then either x sent out a contribution, or x is marked. As x can send out at most 4 contributions, x can receive at most 5 contributions. If x receives 4 contributions, then both $f(x), l(x)$ are marked. If x receives 5 contributions, then $f(x), l(x), x$ are all marked (and by definition, x becomes a dummy vertex).*

Observation 5.5 *For each vertex y , if $f(y)$ is not a dummy vertex, then $f(y)$ received at least $\lceil t(y)/2 \rceil$ contributions from $B(y)$; if $l(y)$ is not a dummy vertex, then $l(y)$ received at least $\lfloor t(y)/2 \rfloor$ contributions from $B(y)$.*

Observation 5.6 *When a vertex in $B(y)$ becomes active, $t(y)$ increases by 1, unless both $f(y)$ and $l(y)$ (if they exist) are dummy vertices.*

In the following, by a *son* of x we mean an in-neighbour of x in G_1 .

Observation 5.7 *Each inactive vertex x has at most $a + 2$ active sons.*

Proof If x is inactive, then x received no contribution from its sons. So if y is a major son of x , then $t(y) = 0$, and if y is a minor son of x , then $t(y) \leq 1$. This means that no major son of x is active, and for each minor son y , $B(y)$ contains at most one active vertex. Therefore x has at most $a + 2$ active sons by Observation 5.2. \square

Observation 5.8 *If a vertex x has k active sons and x is not a dummy vertex, then x received at least $\lceil (k - (a + 2))/2 \rceil$ contributions, and has made at least $\lceil (k - (a + 2))/2 \rceil - 1$ contributions to its parents.*

Proof As in the proof of Observation 5.7, x has at most $a + 2$ minor sons such that when they are activated, they do not make contributions to x . For all the other active sons, at least half of them make contributions to x . As observed above, when x receives one contribution, it will make a contribution to $\{f(x), l(x)\}$ or will be marked. Hence x has made at least $\lceil (k - (a + 2))/2 \rceil - 1$ contributions to its parents. \square

Lemma 5.9 *Assume Alice has just finished a move and x is an unmarked vertex. Then x has at most $(2b + a + 4)\Delta + 5a - b + 66$ active vertices at distance one or two in G .*

Proof Observe that vertices which are at distance one or two from x are contained in at least one of the following vertex sets:

- $f(x)$ and $l(x)$, as well as the vertices adjacent in G to one of these two vertices. There are at most 2Δ such vertices.
- vertices adjacent in G_2 to a vertex in $N_G[x]$ (i.e., the close neighbourhood of x in G). There are at most $b\Delta$ such vertices
- vertices adjacent in G to a neighbour of x in G_2 . There are at most $b(\Delta - 1)$ such vertices.
- vertices in $N_{G_1}^-(x)$ and vertices in $\cup_{y \in N_{G_1}^-(x)} N_{G_1}^-(y)$.

Note that vertices of $\cup_{y \in N_{G_1}^-(x)} N_{G_1}^+(y)$ are included in the counting above, since these vertices belong to $N_G(x) = N_{G_2}(x) \cup N_{G_1}^-(x) \cup \{f(x), l(x)\}$.

By Observation 5.7, every inactive son of x has at most $a + 2$ active sons. Assume x has k active sons y_1, y_2, \dots, y_k . Since x is unmarked, by Observation 5.4, x received at most 4 contributions. Hence $\lceil (k - (a + 2))/2 \rceil \leq 4$ (by Observation 5.8), implying that $k \leq a + 10$. Assume for $i = 1, 2, \dots, k$, y_i has k_i active sons.

By Observation 5.8, y_i made at least $k'_i = \lceil (k_i - (a + 2))/2 \rceil - 1$ contributions to its parents. By Observation 5.5, x received at least $\lfloor \sum_{y_j \in B(y_i)} k'_j/2 \rfloor \geq \sum_{y_j \in B(y_i)} \lfloor k'_j/2 \rfloor$ contributions from $B(y_i)$. By Observation 5.4, $\sum_{i=1}^k \lfloor k'_i/2 \rfloor \leq 4$. This implies that $\sum_{i=1}^k k_i \leq (a + 6)k + 16$.

Hence, there are at most $(a + 6)k + 16$ active sons of an active son of x . So $\cup_{y \in N_{G_1}^-(x)} N_{G_1}^-(y)$ contains at most $(a + 2)(\Delta - k) + (a + 6)k + 16 = (a + 2)\Delta + 4k + 16$ active vertices. In total, $N_{G_1}^-(x)$ and $\cup_{y \in N_{G_1}^-(x)} N_{G_1}^-(y)$ contain at most $(a + 2)\Delta + 5k + 16 \leq (a + 2)\Delta + 5a + 66$ active vertices. Hence, x has at most $(2b + a + 4)\Delta + 5a - b + 66$ active vertices at distance one or two in G . \square

According to the rules and the construction of P_x , marked vertices are all active after Alice's move, and so any unmarked vertex has at most $(2b + a + 4)\Delta + 5a - b + 66$ marked vertices at distance one or two. Hence, after Bob's move, an unmarked vertex has at most $(2b + a + 4)\Delta + 5a - b + 67$ marked vertices at distance one or two in G . This proves that the game colouring number of the square of an (a, b) -pseudo-partial 2-tree with maximum degree Δ is at most $(2b + a + 4)\Delta + 5a - b + 68$.

Since planar graphs are $(3, 8)$ -pseudo-partial 2-trees [12] and partial 2-trees are $(0, 0)$ -pseudo-partial 2-trees, we have the two following corollaries.

Corollary 5.10 *Let G be a planar graph with maximum degree Δ , then $col_g(G^2) \leq 23\Delta + 75$.*

Corollary 5.11 *Let G be a partial 2-tree with maximum degree Δ , then $col_g(G^2) \leq 4\Delta + 68$.*

6 Acyclic game chromatic number

Acyclic game colourings of graphs were recently studied in [4]. This colouring game is the same as the colouring game defined in the introduction, except that at any step, the partial colouring has to be acyclic (that is, a proper colouring without bi-coloured cycles). The

acyclic game chromatic number of a graph G is denoted by $\chi_{a,g}(G)$, and is defined as the least number of colours for which Alice has a winning strategy in G . Surprisingly, while the acyclic chromatic number of planar graphs is at most 5 [3], their acyclic game chromatic number is not bounded. An example of a partial 2-tree with acyclic game chromatic number at least $\Delta/2$ was given in [4]. In general, obtaining good upper bounds for the acyclic game chromatic number seems difficult. The following observation connects acyclic game colouring and the topic of the present paper:

Observation 6.1 *For every graph G , $\chi_{a,g}(G) \leq \text{col}_g(G^2)$.*

If Alice has a strategy to win the marking game in G^2 with k colours, then by using the same strategy she can win the acyclic game with k colours. When playing, Alice picks a vertex v such that at any step of the game, any unmarked vertex has at most $k - 1$ marked vertices at distance one or two. She then colours v with a colour distinct from all the colours at distance at most two from v . She eventually obtains a proper colouring of G^2 , which is also an acyclic colouring of G .

As corollaries of Observation 6.1, we obtain that planar graphs with maximum degree Δ have acyclic game chromatic number at most $23\Delta + 75$, and partial 2-trees with maximum degree Δ have acyclic game chromatic number at most $4\Delta + 68$.

We conclude with some open questions:

Question 6.2 *Is it true that $\chi_{a,g}(G) \leq \chi_g(G^2)$ for every graph G ?*

Question 6.3 *Is it true that for some constant C_1 , any planar graph G with maximum degree Δ satisfies $\chi_{a,g}(G) \leq \frac{\Delta}{2} + C_1$?*

Question 6.4 *Is it true that for some constants C_2 and C_3 , any outerplanar graph G with maximum degree Δ satisfies $\text{col}_g(G^2) \leq \Delta + C_2$, and any planar graph G with maximum degree Δ satisfies $\text{col}_g(G^2) \leq \frac{3}{2}\Delta + C_3$?*

References

- [1] T. Bartnicki, B. Brešar, J. Grytczuk, M. Kovše, Z. Miechowicz, and I. Peterin. *Game chromatic number of cartesian product graphs*, Electronic J. Combin. **15** (2008), #R72.
- [2] H.L. Bodlaender, *On the complexity of some coloring games*, Internat. J. Found. Comput. Sci. **2** (1991), 133–147.
- [3] O.V. Borodin, *On acyclic colorings of planar graphs*, Discrete Math. **25** (1979), 211–236.
- [4] H.-Y. Chang and X. Zhu, *Colourings games on outerplanar graphs and trees*, Discrete Math., in press.
- [5] T. Dinski and X. Zhu, *A bound for the game chromatic number of graphs*, Discrete Math. **196** (1999), 109–115.
- [6] U. Faigle, U. Kern, H.A. Kierstead and W.T. Trotter, *On the game chromatic number of some classes of graphs*, Ars Combin. **35** (1993), 143–150.

- [7] D. Guan and X. Zhu, *The game chromatic number of outerplanar graphs*, J. Graph Theory **30** (1999), 67–70.
- [8] H.A. Kierstead, *A simple competitive graph colouring algorithm*, J. Combin. Theory Ser. B **78** (2000), 57–68.
- [9] H.A. Kierstead and W.T. Trotter, *Planar graph colouring with an uncooperative partner*, J. Graph Theory **18(6)** (1994), 569–584.
- [10] J. Wu and X. Zhu, *Lower bounds for the game colouring number of planar graphs and partial k -trees*, Discrete Math. **308** (2008), 2637–2642.
- [11] X. Zhu, *The Game Coloring Number of Planar Graphs*, J. Combin. Theory Ser. B **75** (1999), 245–258.
- [12] X. Zhu, *The game coloring number of pseudo partial k -trees*, Discrete Math. **215** (2000), 245–262.
- [13] X. Zhu, *Refined activation strategy for the marking game*, J. Combin. Theory Ser. B **98** (2008), 1–18.
- [14] X. Zhu, *Game coloring the Cartesian product of graphs*, J. Graph Theory **59** (2008), 261–278.