Flows in graphs and matroids

ICGT Grenoble, 2014

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Defining flows

A *flow instance*:

- Graph $G = (V, E)$
- Demand edges: $\Sigma \subseteq E(G)$
- Capacity edges: $E(G) \setminus \Sigma$
- Integer weights $w_e \geq 0$ for all $e \in E$

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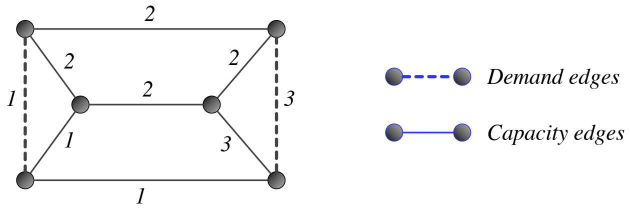

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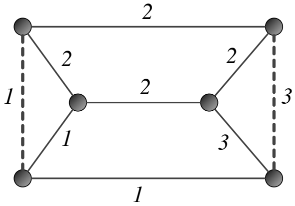

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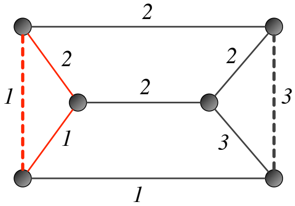

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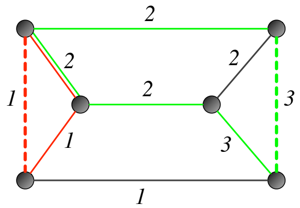

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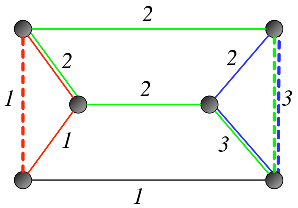

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A necessary condition

Do we always have a flow?

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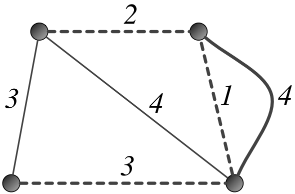

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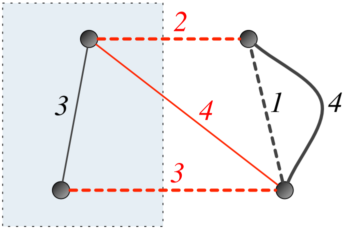

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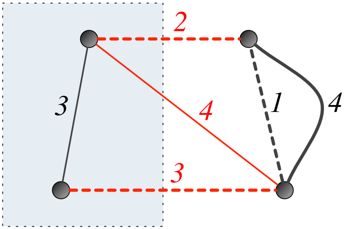

Demand across cut $= 2 + 3$

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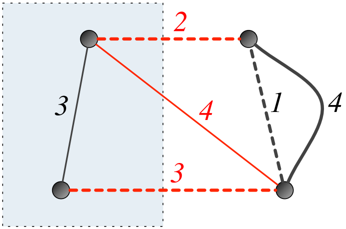

Demand across cut $= 2 + 3$

Capacity across cut $= 4$

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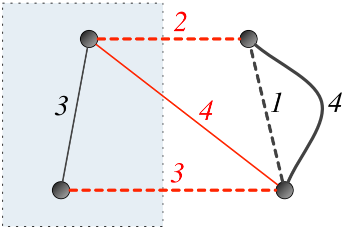

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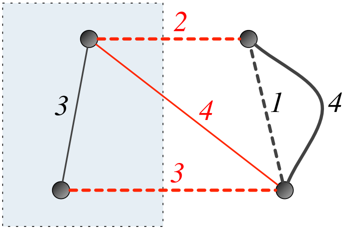

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The cut condition: For every cut:

$$\boxed{\text{Demand across the cut} \quad \leq \quad \text{Capacity across the cut.}}$$

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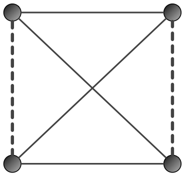

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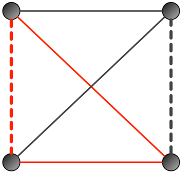

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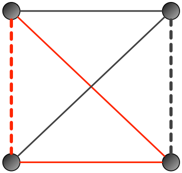

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Theorem [Seymour 1977]

There exists an integer flow if

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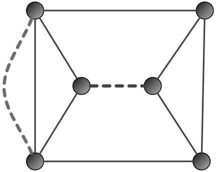

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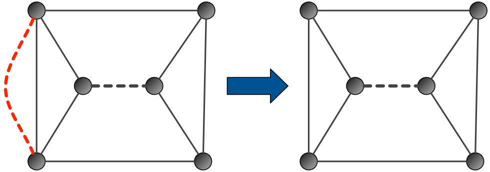

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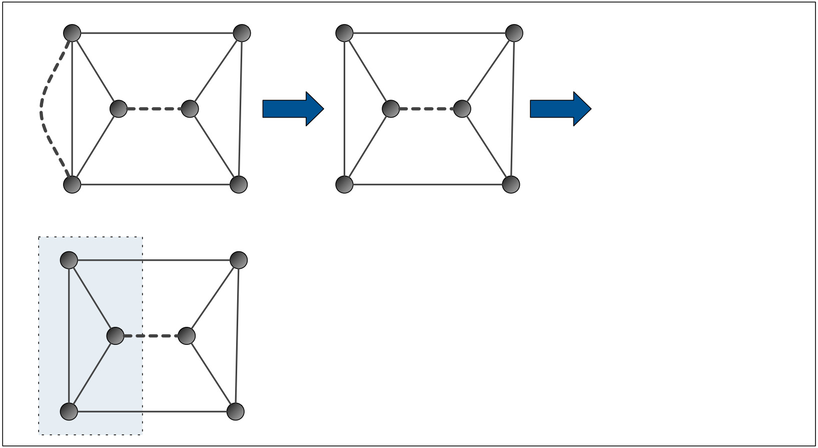

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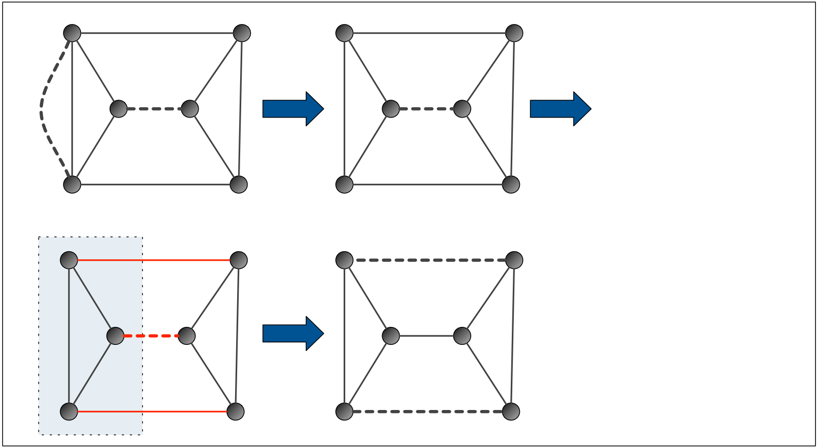

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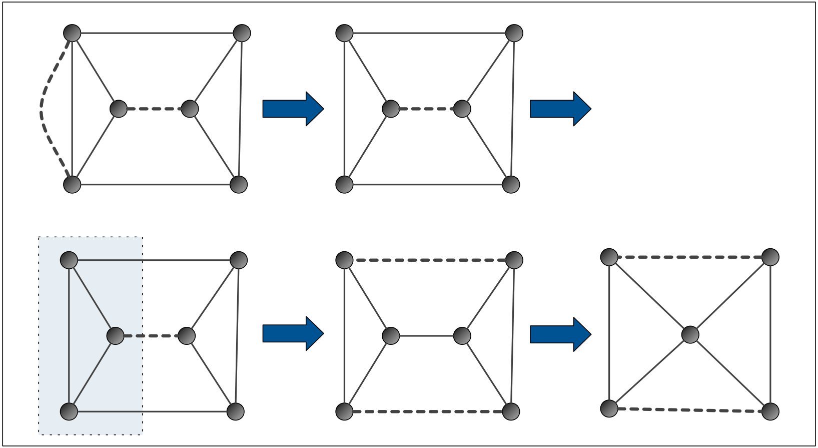

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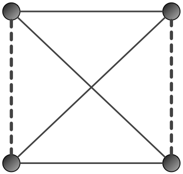

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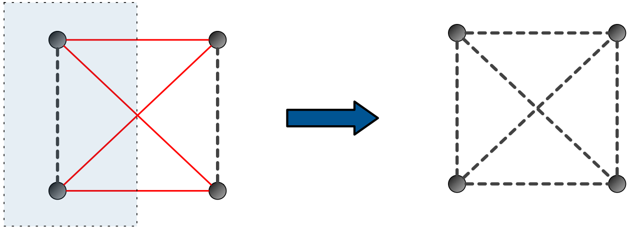

odd-K_4

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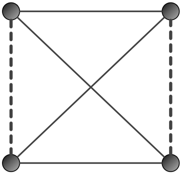
Is this a bad example for fractional flows?



All capacities/demands $= 1$

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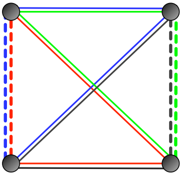

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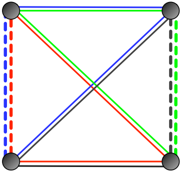

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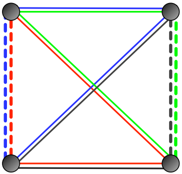

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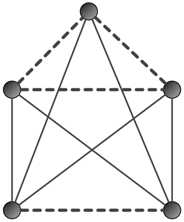

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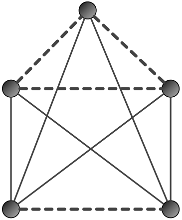

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Total demand $= 4$

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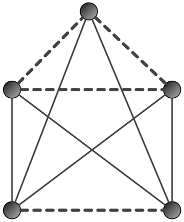

All capacities/demands $= 1$

Total demand $= 4$

Total capacity $= 6$

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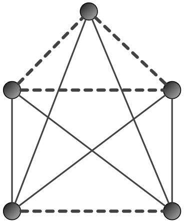

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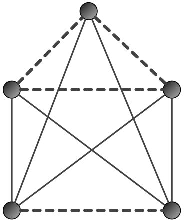

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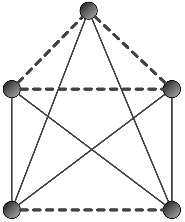

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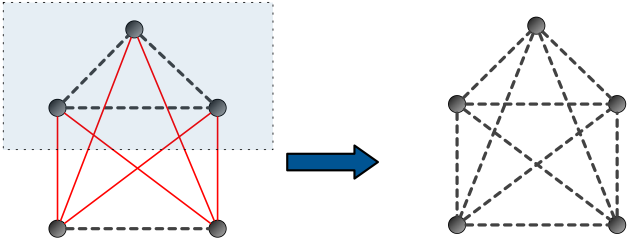

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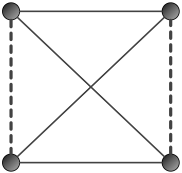

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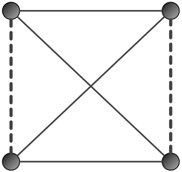

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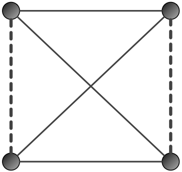

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When is the cut-condition sufficient for the existence of an integer flow for the case of Eulerian demands/capacities?

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Open problem

Find a polytime algorithm with input G, Σ and w Eulerian that returns:

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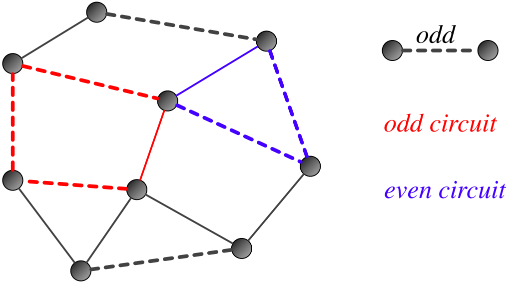
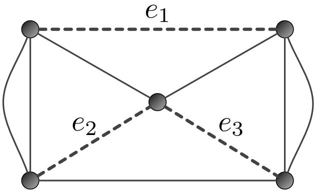


Flow feasibility problems are minimax problems

Flow instance: all capacities/demands $= 1$



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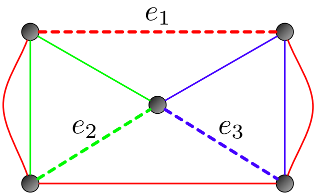

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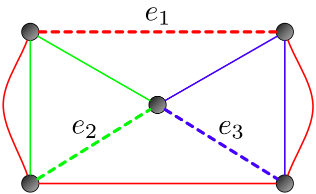

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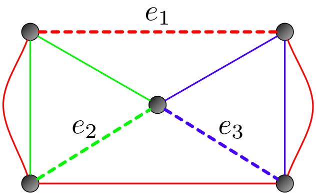

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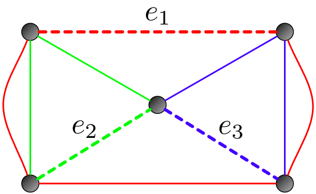

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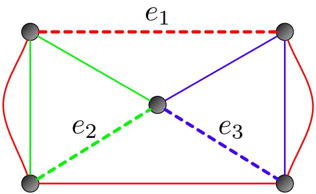

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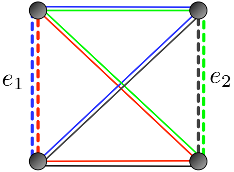

$y_{C_1} = \frac{1}{2}$ $y_{C_2} = \frac{1}{2}$ $y_{C_3} = \frac{1}{2}$ $y_{C_4} = \frac{1}{2}$

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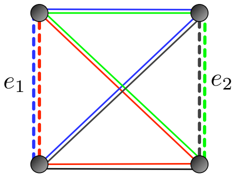

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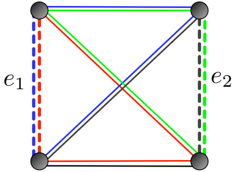

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Restating the theorems

Theorem [Seymour 1977]

There exists an integer flow if

1. the cut-condition holds,
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The odd circuits pack if there is no odd-K_4 minor.

Restating the theorems

Theorem [G 2001]

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Restating the theorems

Theorem [Geelen, G 2002]

Suppose the demand/capacities are Eulerian.
There exists an integer flow if

1. the cut-condition holds,
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Theorem

The odd circuits of an Eulerian graph pack if there is no odd-K_5 minor.

Let (G, Σ) be a signed graph. Then

(a) the odd circuits pack if no odd-K_4 minor.

(b) the odd circuits fractionally pack if no odd-K_5 minor.

(c) the odd circuits pack if no odd-K_5 minor and the graph is Eulerian.

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For weighted version, theorems are "if and only if"

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Question

What can we say if we replace

"signed graphs" by "signed binary matroids"?

Binary matroids - everything you need to know

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$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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- C is a cycle of matroid M_A iff $C \in rowspan(A)^{\perp}$

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- Inclusion-wise minimal cycles are circuits

Binary matroids - everything you need to know

Example: Let G be a graph,

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M_A is co-graphic

Signed binary matroids

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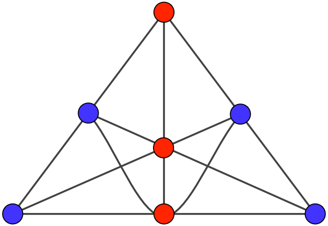

Fano matroid

Σ = set of all elements

odd circuit

even circuit

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elements in Σ are odd the other elements are even

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Minors for (M, Σ):

- Delete $e \in E(G)$,
- Contract $e \notin \Sigma$,
- Resign: replace Σ by $\Sigma \triangle B$ where B is a cocycle.

Packing odd circuits in binary matroids

Recall:

Theorem

For a signed graph.
The odd circuits pack if there is no odd-K_4 minor.

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This extends to,

Theorem [Seymour 1977]

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Fractionally packing odd circuits in binary matroids

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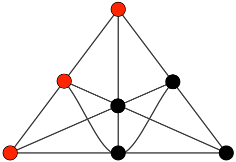

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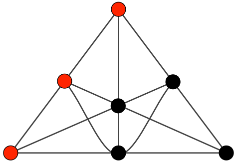

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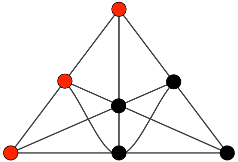

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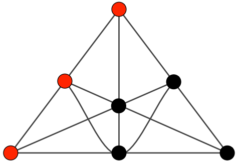

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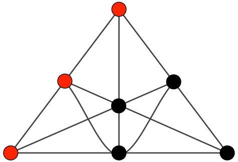

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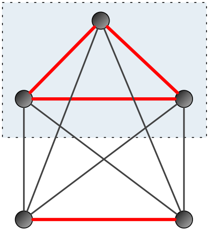

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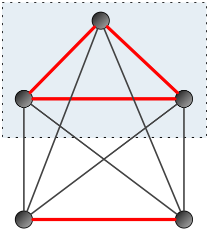

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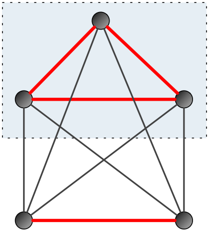

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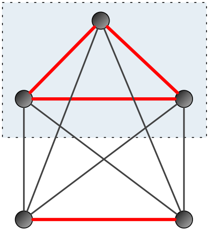

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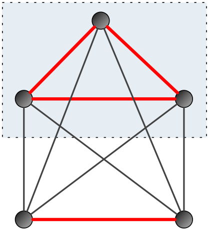

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does not fractionally pack

The Flowing conjecture

Conjecture [Seymour 1977]

The odd circuits of a signed matroid fractionally pack if it does not have any of the following minors:

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$5000 cash prize !!! (Gérard Cornuéjols)

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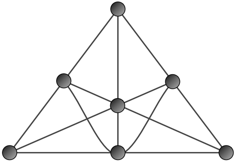
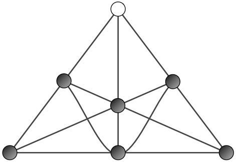
There are only 3 non-bipartite signing of the Fano:

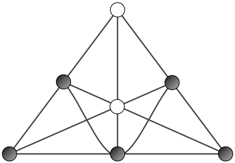
● in Σ ○ not in Σ



Good Fano



Bad Fano Type I



Bad Fano Type II

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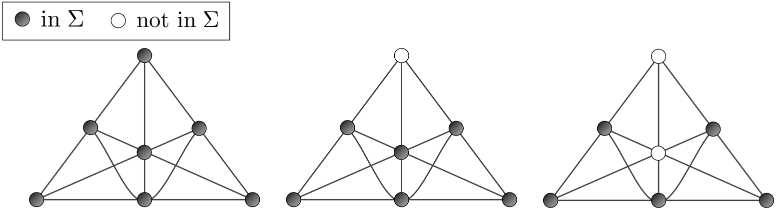

Good Fano *Bad Fano Type I* *Bad Fano Type II*

Theorem [Cornuéjols, G 2002]

Suppose conjecture is false.
Then there exists a counterexample that has a Bad Fano as a minor.

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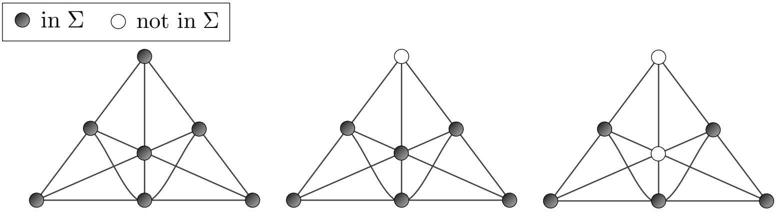

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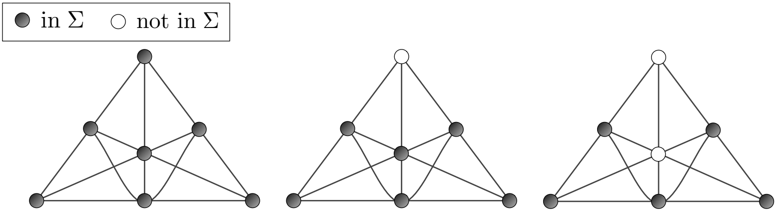

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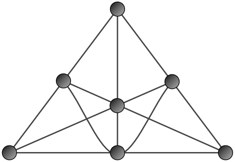
What do we mean by "many"?



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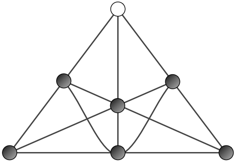

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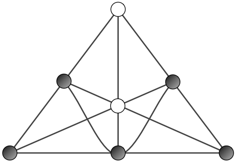
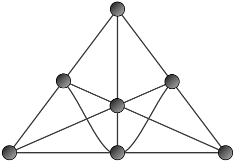


Bad Fano Type II



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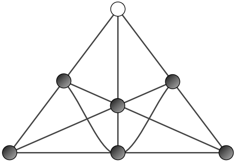

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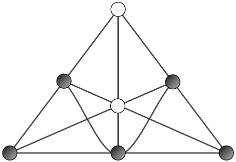

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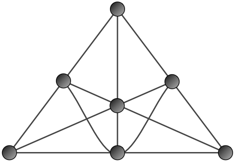

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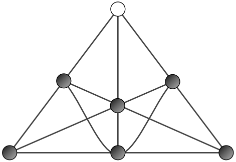

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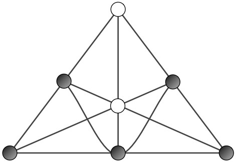

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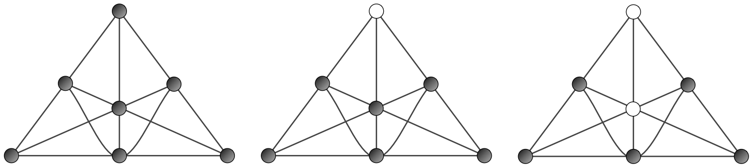

Good Fano *Bad Fano type I* *Bad Fano Type II*

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Suppose conjecture is false.
Then there exists a counterexample (M, Σ) such that,

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Many bad Fanos + high connectivity. Does this imply we have a good Fano?

Packing odd circuits in Eulerian binary matroids

Recall:

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For an Eulerian signed graph.
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What do we need to excluded for the odd circuits to pack:

- odd-K_5,
- the lines of the Fano,
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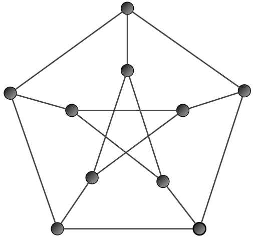

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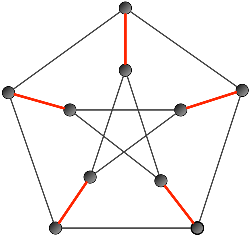

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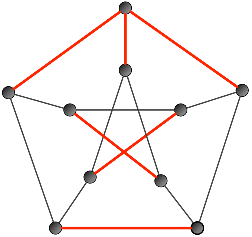

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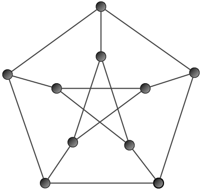

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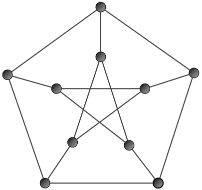

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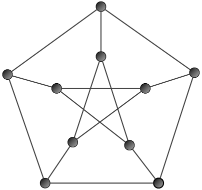

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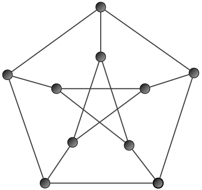

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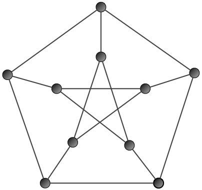

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The Cycling conjecture

Conjecture [Seymour 1981]

The odd circuits of a signed Eulerian matroid pack if it does not have any of the following minors:

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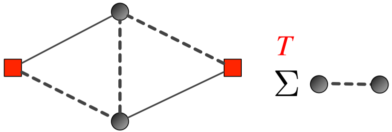

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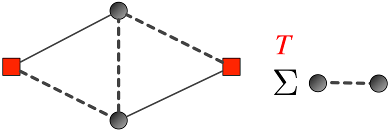

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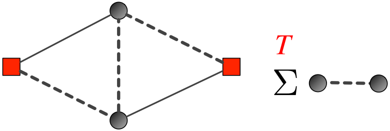

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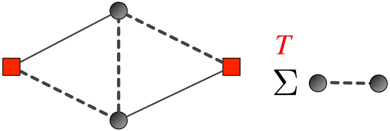

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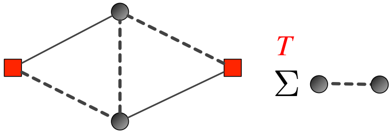

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Theorem [Seymour]

In a bipartite graph the size of the minimum T-join is equal to the maximum number of pairwise disjoint T-cuts.

The Cycling conjecture is really, really hard :(

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A consequence of the Cycling conjecture (if true):

Cubic graphs with no Petersen minors are 3-edge colourable

Implies the 4-colour theorem

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Suppose $(G, E(G))$ has no odd-K_5 minor. If the length of the shortest odd cycle is k, then there exists cuts $B_1, \ldots, B_k$ such that every edge e is in at least $k-1$ of $B_1, \ldots, B_k$.

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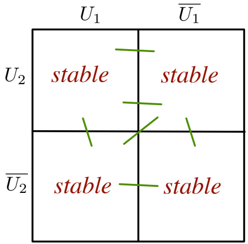

Implies the 4-colour theorem

What to do next

Cycling conjecture holds for

- graphic matroids,
- co-graphic matroids,
- implies the 4-colour in general.

Question

Interesting cases that does not imply the 4-colour theorem?

Finding some interesting classes to study

Definition

Let M be binary matroid, $e \in E(M)$.
An e-path is a set of the form $C - \{e\}$ where C is a circuit of M.

M graphic matroid of G with $e = (s, t)$

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Idea

Find classes of matroids generalizing graphic and co-graphic and study the cycling conjecture for the e-paths of these matroids.

Even cycle matroids

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Then M_A is graphic matroid of G

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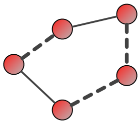
Circuits of M_A:



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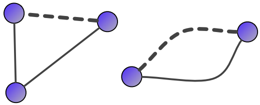

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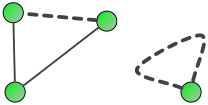

e-paths of even cycle matroid

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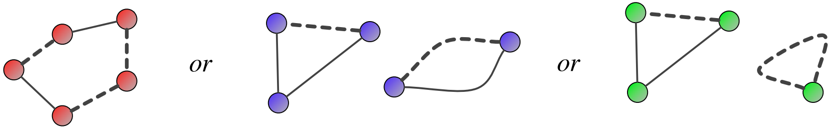

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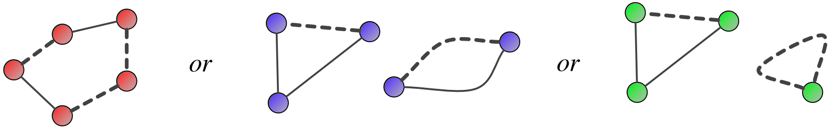

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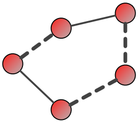

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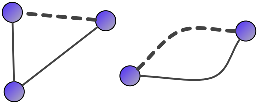

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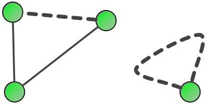

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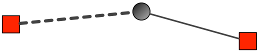

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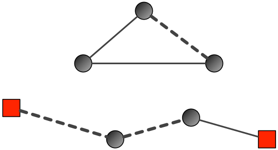

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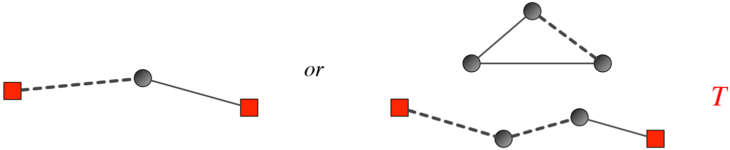
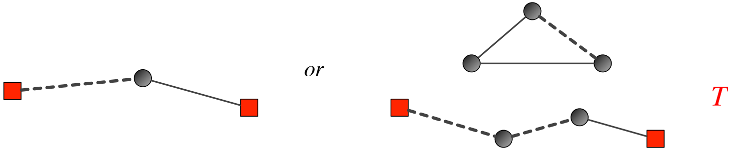


e-paths of even cycle matroid

Assume $e \in \Sigma$.

- If e is loop, then e-paths $=$ odd circuits
- If e not loop, then e-paths are



Thus,

e-paths of even cycle matroids $=$
odd T-joins of signed graph with $|T| \leq 2$.

A special case of the Cycling conjecture

Theorem [Abdi, G 2014]

The Cycling conjecture holds for e-paths of even cycle matroids.

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Let (G, Σ) be a signed graph and $|T| \leq 2$. The odd T-joins pack if

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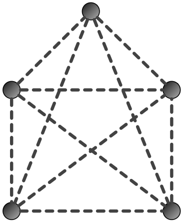
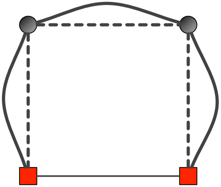


odd-K_5 $(T = \emptyset)$



Line of Fano $(|T| = 2)$

The Eulerian condition

The Eulerian condition for (G, Σ) and $|T| \leq 2$ says,

- if $v \notin T$ then degree of v is even,
- if $v \in T$ then parity of degree of v is same as parity of $|\Sigma|$.

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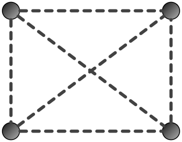
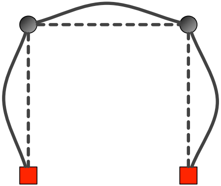
Is it really needed? YES



$T = \emptyset$

Does not pack



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Special cases

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Question

Any interesting special cases? MANY

Special case: Packing odd cycles

$\boxed{\text{Special case: } T = \emptyset}$

Theorem [Geelen,G 2002]

For an Eulerian signed graph.
The odd circuits pack if there is no odd-K_5 minor.

Special case: Packing T-joins

$\boxed{\text{Special case: } (G,\Sigma) \setminus v \text{ bipartite for } v \notin T}$

$\Longrightarrow$

Theorem

Let G be a graph and $|T| \leq 4$. Suppose vertices not in T have even degree and vertices in T have degrees of the same parity. Then the minimum size of a T-cut is equal to the maximum number of pairwise disjoint T-joins.

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Holds for $|T| \leq 8$ (Cohen 97)

Special case: 2-commodity flow

$\boxed{\text{Special case: } T = \{s,t\},\ (G,\Sigma) \setminus \{s,t\} \text{ bipartite}}$

$\Longrightarrow$

Theorem [Hu 63, Rothschild, Whinston 66]

Let G be a graph with vertices s_1, t_1, s_2, t_2. Suppose all of s_1, t_1, s_2, t_2 have the same degree parity and all the other vertices have even degree. Then the minimum number of edges needed to intersect all $s_i t_i$ paths equals the maximum number of pairwise disjoint $s_i t_i$-paths $(i = 1, 2)$.

Special case: covering with cuts

$\boxed{\text{Special case: } G \text{ plane graph}}$

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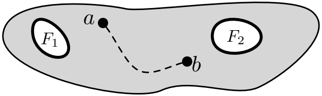

Special case: covering with cuts

Special case: G plane graph

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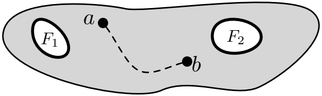

Theorem

Let G as in $(\star)$. If the length of the shortest odd cycle is k, then there exists cuts $B_1, \ldots, B_k$ such that every edge e is in at least $k-1$ of $B_1, \ldots, B_k$.

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Thus we proved the following conjecture for graphs of type $(\star)$!!!

Conjecture

Suppose $(G, E(G))$ has no odd-K_5 minor. If the length of the shortest odd cycle is k, then there exists cuts $B_1, \ldots, B_k$ such that every edge e is in at least $k-1$ of $B_1, \ldots, B_k$.

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Open problem

Prove the conjecture for the class of graphs obtained from a plane graph with exactly two odd faces by adding an apex vertex.

Special cases: topological classes

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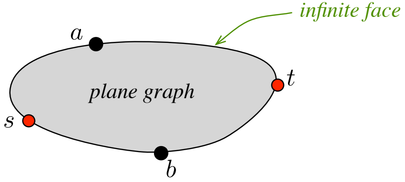
Example:

$T = \{s, t\}$

G is a plane graph

$(G, \Sigma) \setminus \{a, b\}$ bipartite



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Example:

$T = \{s, t\}$

(s, t) is an edge

(G, Σ) has embedding on projective plane where every face is even

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Thank you for your attention