Extremal graph theory and lower bounds in distributed computing

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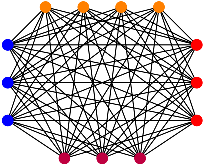
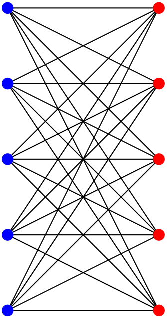


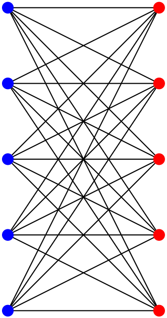
Figure 1: $T(10,4)$

F-free graph: a graph that does not contain F as a subgraph



A bipartite graph contains no odd cycles, is therefore $C_{2\ell+1}$-free.

F-free graph: a graph that does not contain F as a subgraph



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Theorem (Mantel, 1907)

The balanced complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ *maximizes the number of edges of triangle-free graphs.*

For a graph F, the Turán number $ex(n, F)$ is the largest number of edges in an n-vertex F-free graph.

Theorem (Turán, 1941)

The Turán graph $T(n, r)$ maximizes the number of edges of K_{r+1}-free graphs, i.e.

$$ex(n, K_{r+1}) = e(T(n, r)).$$

The Turán graph $T(n, r)$ is a *balanced* complete r-partite graph.

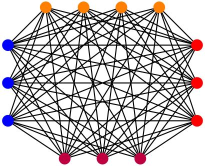

Figure 2: $T(10, 4)$

F-free graphs for general F

A graph F has chromatic number $\chi(F)$ if for large enough n, F is a subgraph of $T(n, \chi(F))$, and $\chi(F)$ is minimal with this property.

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Theorem (Erdős-Stone-Simonovits, 1946/1966)

If F is not bipartite, then

$$ex(n, F) = (1 + o(1))e(T(n, \chi(F) - 1)).$$

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Conclusion: ex(n,F) is well understood except if F is a bipartite graph

Why should we care about the Turán number of bipartite graphs?

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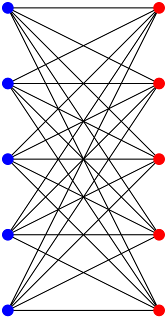
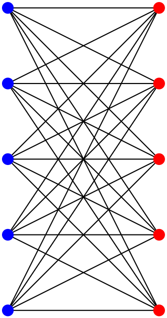
Today: Even cycles and cycle detection



Theorem

For $\ell \geq 1$,
$$ex(C_{2\ell+1}) = e(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) \approx n^2/4$$



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Conjecture

For every $\ell \geq 2$, $ex(n, C_{2\ell}) = \theta(n^{1+\frac{1}{\ell}})$.

$k = 2$: Even cycles

Theorem (Bondy, Simonovitz, 1974)

For every integer $\ell \geq 2$, there exists a constant c so that

$$ex(n, C_{2\ell}) \leq cn^{1+\frac{1}{\ell}}.$$

This is tight for $\ell = 2, 3, 5$ ($\ell = 3, 5$ Benson, Singleton, 1966).

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Wengers construction : $\ell = 3, 5$ uses bipartite point-line incidence graphs of F_q^ℓ

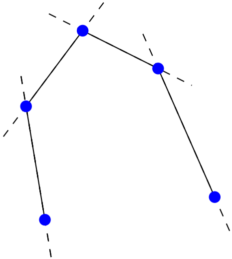
Counting edges: q^ℓ lines and points → $n = 2q^\ell$ and $m = q^{\ell+1}$ (q points per line)

Wenger construction: C_k-freeness

Point-line incidence graph

- points F_q^ℓ, q a prime power,
- lines with direction $(1, t, \dots, t^{\ell-1})$ for some $t \in F_q$.

A cycle $C_{2\ell}$ is of the form $P_1 L_1 P_2 L_2 \dots P_\ell L_\ell P_1$.



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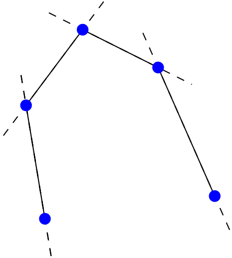

$$\sum_{i=1}^{\ell} a_i(1, t_i, \dots, t_i^{\ell-1}) = 0$$

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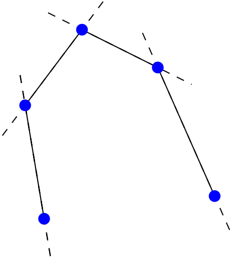


$$\sum_{i=1}^{\ell} a_i(1, t_i, \dots, t_i^{\ell-1}) = 0$$

By Vandermonde matrix, the set of line directions is linearly independent. Each direction has to appear at least twice and not consecutively, not possible for $\ell = 3, 5$.

Best Known General Lower Bound for Even Cycles

Theorem (Lazebnik, Ustimenko, Woldar, 1995)

For every fixed $\ell \geq 2$,

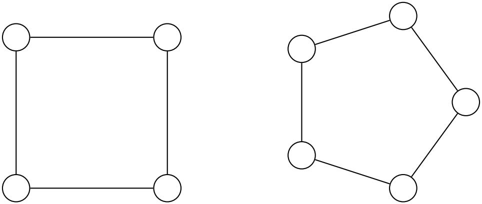
$$\mathrm{ex}(n, C_{2k}) = \Omega\left(n^{1+\frac{2}{3\ell-3}}\right).$$

Construction. Algebraic graphs arising from finite-field incidence structures.

Compare with the upper bound. Bondy–Simonovits proved

$$\mathrm{ex}(n, C_{2\ell}) = O\left(n^{1+\frac{1}{\ell}}\right).$$

Cycle detection

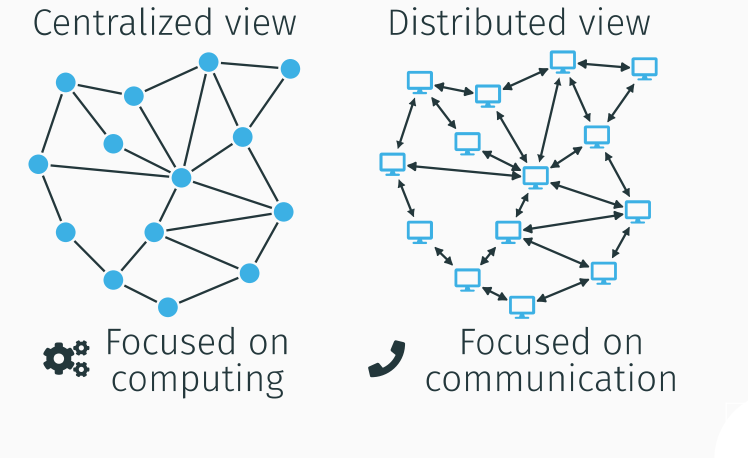


In the centralized view:
To detect a cycle C_ℓ a brute force algorithm needs $O(n^\ell)$ time.

There are FPT algorithms known running in $O(2^\ell)n^{O(1)}$ time.

Distributed Algorithms

Image credit: Timothé Picavet



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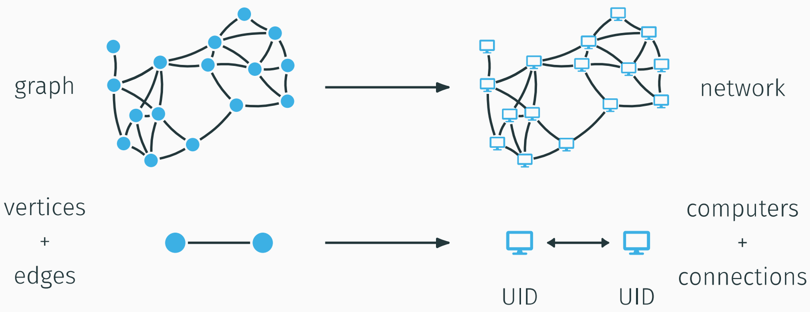


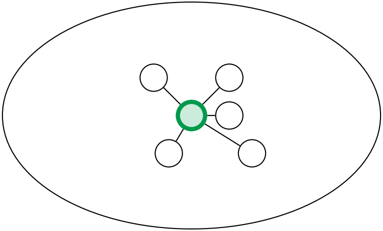
graph → network

vertices + edges → computers + connections

UID UID

Communication

Image credit: Timothé Picavet



- **Round** 0: each node knows only its neighbours.

Communication

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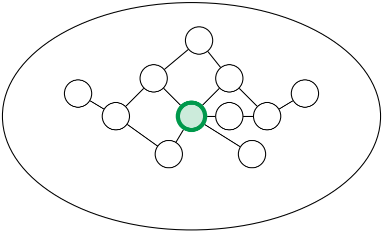

- **Round** 0: each node knows only its neighbours.
- **Round** k: a node can receive messages from distance $\leq k+1$

Communication

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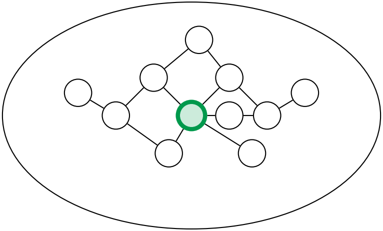
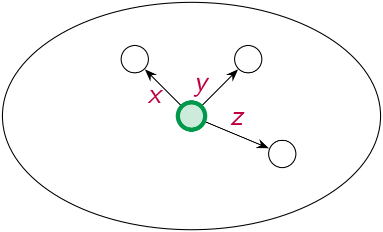


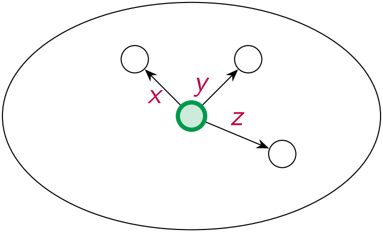
- **Round** 0: each node knows only its neighbours.
- **Round** k: a node can receive messages from distance $\leq k+1$
- **Assumptions:** nodes have unique identifiers and infinite computing power

Complexity of an algorithm = number of communication rounds of the algorithm

LOCAL, CONGEST-unicast, and CONGEST-broadcast

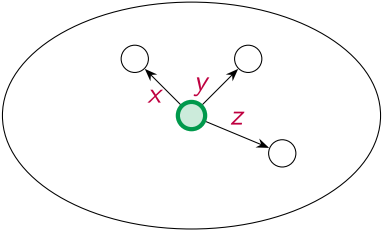


LOCAL, CONGEST-unicast, and CONGEST-broadcast



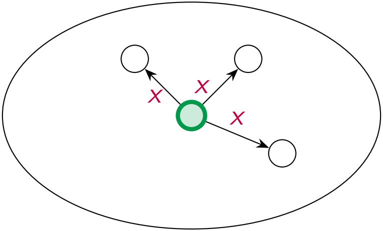
LOCAL: no restriction

LOCAL, CONGEST-unicast, and CONGEST-broadcast



CONGEST-unicast: each message x,y,z with $\leq b$ bits

LOCAL, CONGEST-unicast, and CONGEST-broadcast



CONGEST-broadcast: each message x=y=z with $\leq b$ bits

LOCAL, CONGEST-unicast, and CONGEST-broadcast

LOCAL	**CONGEST-Unicast**	**CONGEST-Broadcast**
Unlimited message size	b bits per edge per round	
		The same message must be sent to all neighbors

Hierarchy:

$$\text{LOCAL} \supseteq \text{CONGEST-Unicast} \supseteq \text{CONGEST-Broadcast}.$$

How many communication rounds needed to detect C_ℓ in the LOCAL model?

Main Result

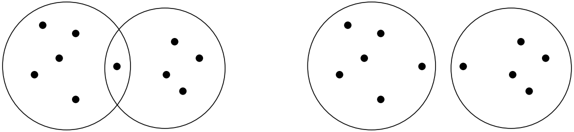
Theorem (Informal)

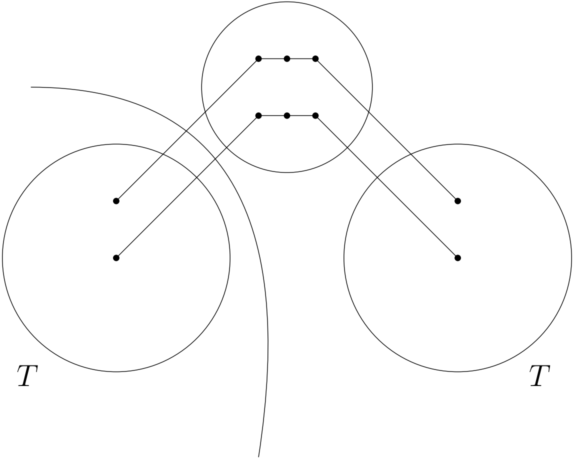
Detecting C_ℓ ($\ell \geq 4$) in CONGEST requires polynomially many rounds.

Proof Sketch

1. Encode Set Disjointness into two edge sets.
2. If the inputs are disjoint, the graph is C_ℓ-free.
3. If the inputs intersect, a cycle C_ℓ appears.

Set Disjointness

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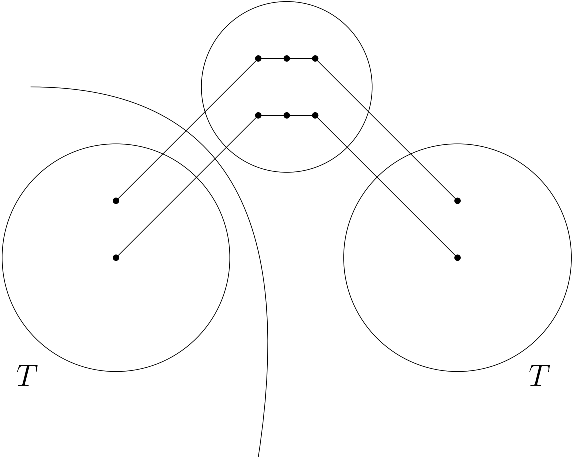


Theorem (Drucker, Kuhn, Oshman, 2014)

For $\ell \geq 2$, $C_{2\ell}$- detection requires $\Omega(ex(n, C_{2\ell})/(nb))$ rounds
CONGEST-unicast.

An improved lower bound for even cycles:

Theorem (Korhonen, Rybicki, 2017)

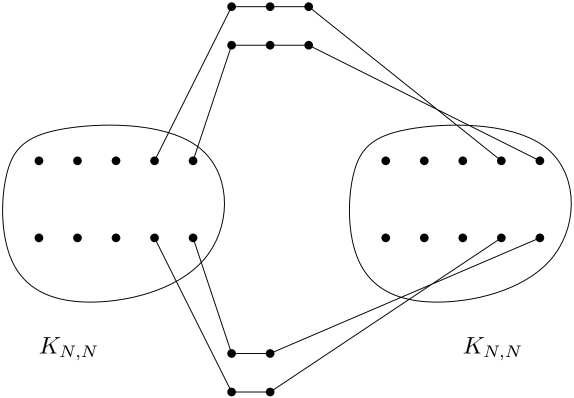
For $\ell \geq 2$,$C_{2\ell}$*- detection requires* $\Omega(\sqrt{n}/b)$ *rounds in CONGEST-unicast.*

The best upper bound:

Theorem (Fraigniaud, Luce, Magniez, and Todinca, 2025)

For $\ell \geq 2$, $C_{2\ell}$*-detection can be solved in* $O(n^{1-1/\ell})$ *rounds in CONGEST-broadcast.*

Construction for odd cycles:



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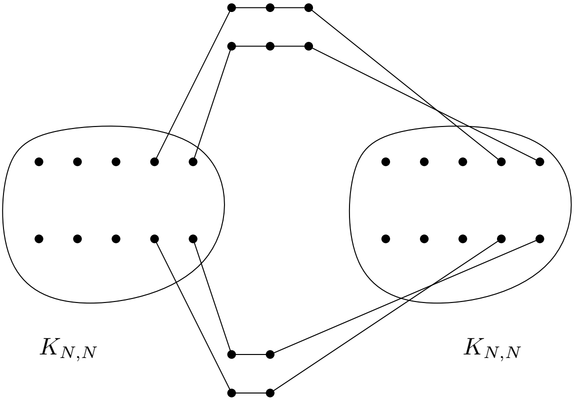


Rounds needed: $\Omega(ex(n, C_{2\ell+1})/(nb)) = \Omega(n/b)$

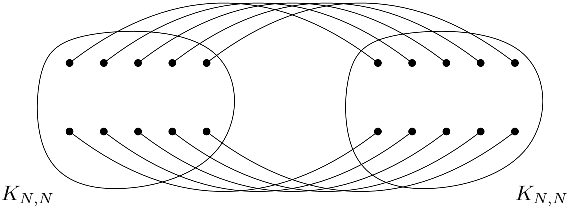
Theorem (Drucker, Kuhn, Oshman, 2014)

For every fixed $\ell \geq 2$, $C_{2\ell+1}$*- detection requires at least* $\Omega(ex(n, C_{2\ell+1})/(nb)) = \Omega(n/b)$ *rounds in CONGEST-unicast.*

Theorem (Korhonen, Rybicki, 2017)

For every $\ell \geq 2$, $C_{2\ell+1}$*-detection can be solved in* $O(n)$ *rounds in CONGEST-broadcast.*

A better lower bound for even cycles



Thank you!

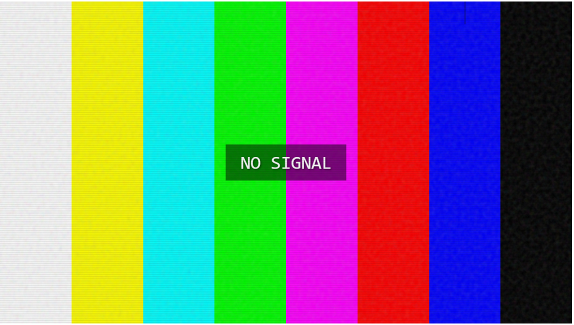


Figure 3: No broadcast

Drucker, Kuhm, Oshman: On the Power of the Congested Clique Model