DEGENERACY AND LIST COLORING

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Abstract. Let G be a d-degenerate graph. Assume that each vertex has a list of $d+1$ colors. Then G has a proper coloring in which every vertex receives a color from its own list. It was proved by Dvořák, Norin, and Postle that surprisingly, this remains true even if an arbitrary color is removed from the list of an arbitrary vertex. Their proof uses the polynomial method. The purpose of this short note is to describe an alternative, purely combinatorial proof, which was inspired by the combinatorial proof of the Alon-Tarsi theorem obtained by Schauz and was suggested by ChatGPT Sol 5.6.

1. Introduction

Given a graph G and an integer k, a *k-list-assigment* to G is a function $L : V(G) \to 2^{\mathbb{N}}$ such that for every vertex $v \in V(G)$, $|L(v)| \geqslant k$. An *L-coloring* of G is a coloring c of the vertices of G in which every vertex receives a color from its own list, that is $c(v) \in L(v)$ for any $v \in V(G)$. The graph G is said to be *L-colorable* if G has a proper L-coloring.

A graph is *d-degenerate* if any subgraph H of G contains a vertex of degree at most d in H. Equivalently, $V(G)$ can be ordered such that every vertex has at most d neighbors preceding it in the order. A classical result in graph theory states that for any d, d-degenerate graphs are L-colorable for any $(d+1)$-list-assignment L. This follows from a simple greedy coloring procedure in the degeneracy order. The resulting coloring can therefore be obtained constructively and efficiently. This result is best possible in the sense that $d+1$ cannot be replaced by d (consider for instance a complete graph on $d+1$ vertices). However, there is still some room for improvement if we are allowed to decrease the size of individual lists. The following result was proved in [DNP19].

Theorem 1 ([DNP19])**.** *Let $d \geqslant 2$ be an integer, let G be a d-degenerate graph, and let L be a $(d+1)$-list-assignment to G. Let L' be obtained from L by removing an arbitrary color from the list of an arbitrary vertex. Then G is L'-colorable.*

This theorem does not appear explicitly in [DNP19]. Dvořák, Norin, and Postle proved that when $d+1$ is prime, a stronger result holds: d colors can be arbitrarily removed from individual lists (instead of a single color) while still guaranteeing the existence of a list-coloring. The proof follows the same lines, and requires an application of the Combinatorial Nullstellensatz [Alo99]. Note that the two results are incomparable, as Theorem 1 does not require $d+1$ to be a prime number.

I have been trying to find a constructive proof of this theorem for some time (the Combinatorial Nullstellensatz only proves the existence of the coloring), or at least a combinatorial proof (instead of an algebraic one), and discussed the problem with a number of students and colleagues. In the summer 2026, while experimenting with the latest commercial large language models to assess their strengths and weaknesses, I asked them to work on a number of open problems on which I had been working over the years. This was an almost complete failure, except for this list coloring problem. ChatGPT Sol 5.6 found that an answer to my questions already existed in the literature: Theorem 1 could be proved using the Alon-Tarsi theorem [AT92] rather than by a direct application of

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the Combinatorial Nullstellensatz, while Schauz [Sch10] had given a purely combinatorial proof of the Alon-Tarsi theorem. More importantly, ChatGPT Sol 5.6 found that the approach of Schauz, specialized to this list-coloring problem, gave a very nice and simple proof of Theorem 1. The purpose of this short note is to record this alternative proof, in case it is helpful for anyone. The original paper [DNP19] initiated a significant body of subsequent work and generalizations, in particular list-coloring packing [CCvBDK24], and I hope that the simple ideas in the present note trigger further developments.

2. The proof

A *weighted graph* is a pair (G, w) where G is a graph and $w : V(G) \to \mathbb{Z}$ is a weight function. The *w-weight* (or simply *weight*, if w is clear from the context) of a subset $S \subseteq V(G)$ is defined as $\sum_{v \in S} w(v)$. The weight of a color c in some coloring of G is the weight of the set of vertices colored with color c. We say that a weight function w of G is *L-forcing* if for every color c in the lists, the weight of c is the same in all L-colorings of G. Note that in this definition, different colors can have different weights.

What kind of weight functions are L-forcing? A simple example is the uniform 0 weight. Then every color has weight 0 in all colorings. Here is a more interesting example: consider the case $G = K_{d+1}$ and assume all vertices of G have the same list of size $d + 1$ and are assigned weight 1. Then every color has weight 1 in all L-colorings. These two examples can be mixed in a natural way, assigning weight 1 to all vertices in disjoint copies of K_{d+1}, and 0 to all the other vertices. In these examples, the total weight is always a multiple of $d + 1$. We prove that this is a necessary condition.

Lemma 2. *Let $d \geqslant 2$ be an integer. Let (G, w) be a d-degenerate weighted graph, together with a $(d + 1)$-list-assignment L. If w is L-forcing, then the weight of $V(G)$ is divisible by $d + 1$.*

Proof. We prove the result by induction on the number of vertices of G. Suppose that w is L-forcing, and let v be a vertex of degree at most d in G. Note that any L-coloring of $G - v$ extends to G. If $w(v) = 0$, then w is also L-forcing in $G - v$. By the induction hypothesis, $w(G-v) = 0 \pmod{d+1}$, and thus $w(G) = 0+0 = 0 \pmod{d+1}$. So we can assume in the remainder that $w(v) \neq 0$. This implies in particular that every L-coloring f of $G - v$ extends to G in a unique way. This means that all d colors of $L(v) - f(v)$ appear exactly once on $N(v)$. Consider the following weight function on $G - v$:

$$w'(x) = \begin{cases} w(x) - w(v) \text{ if } x \in N(v) \\ w(x) \text{ otherwise.} \end{cases}$$

If $c \in L(v)$, then the w'-weight of c in an L-coloring f of $G - v$ is the w-weight of c in the unique extension of f to G, minus $w(v)$ (either c appears in $N(v)$, or on v). By assumption, the w-weight of c is independent of f, and thus the w'-weight of c is also independent of the coloring in $G - v$. If $c \notin L(v)$, then the w'-weight of c in an L-coloring f of $G - v$ is equal to its w-weight in G, which is independent of f. It follows that $(G - v, w')$ satisfies the assumptions of the lemma. By the induction hypothesis, $w'(V(G) - v)$ is divisible by $d + 1$. As

$$w'(V(G) - v) = w(V(G)) - (d + 1) \cdot w(v),$$

it follows that $w(V(G))$ is also divisible by $d + 1$, as desired. □

We now deduce Theorem 1 from Lemma 2.

Proof of Theorem 1. We prove the result by induction on the number of vertices. Let v be the vertex such that $|L'(v)| = d$ and all vertices $u \neq v$ have $|L'(u)| \geqslant d + 1$ (if $|L'(v)| \geqslant d + 1$, then the result holds trivially).

If some vertex $u \neq v$ satisfies $d(u) \leqslant d$, then we remove u, apply induction, and assign to u any color distinct from the colors of all its neighbors. Such a color exists, since $|L(u)| > d(u)$. If $d(v) < d$ we can remove v, apply induction, and assign to v any color distinct from the colors of all its neighbors. Such a color exists, since $|L(v)| > d(v)$. So we can assume in the remainder that $d(v) = d$ and $d(u) > d$ for each $u \neq v$. As $G - v$ contains a vertex of degree at most d and all vertices of G distinct from v have degree at least $d + 1$ in G, v has a neighbor u such that $d_{G-v}(u) = d$.

(1) For every L'-coloring f of $G - v$, all d colors of $L'(v)$ appear on $N_G(v)$, since otherwise we can color v greedily. In particular, all $d - 1$ colors of $L'(v) - f(u)$ appear on $N_G(v) - u$.
(2) For every L'-coloring f of $G - \{u, v\}$, there is a unique choice for $f(u) \in L'(u)$, so all d colors of $L'(u) - f(u)$ appear once on $N_{G-v}(u)$.

For a subset S of some ground set Ω, consider the indicator function $\mathbf{1}_S : \Omega \to \mathbb{Z}$ defined as:

$$\mathbf{1}_S(x) = \begin{cases} 1 \text{ if } x \in S \\ 0 \text{ otherwise.} \end{cases}$$

It will be convenient to use indicator functions both for subsets of vertices of G and for subsets of colors.

Define the weight function $w = \mathbf{1}_{N_{G-v}(u)} - \mathbf{1}_{N_G(v)-u}$ in $G - \{u, v\}$. We now argue that we are in a position to apply Lemma 2 to $(G - \{u, v\}, w)$.

Fix some L'-coloring f of $G - \{u, v\}$, and let $f(u)$ be the color of u in the unique extension of f to an L'-coloring of $G - v$. Let c be an arbitrary color. By (1) and (2), the weight of c in $G - \{u, v\}$ with respect to the coloring f is equal to

$$\mathbf{1}_{L'(u)-f(u)}(c) - \mathbf{1}_{L'(v)-f(u)}(c) = \mathbf{1}_{L'(u)}(c) - \mathbf{1}_{L'(v)}(c).$$

As this weight is independent of f, and this holds for any color c, it follows from Lemma 2 that the weight of $V(G) - \{u, v\}$ is divisible by $d + 1$. However,

$$w(V(G) - \{u, v\}) = |L'(u)| - |L'(v)| = (d + 1) - d = 1,$$

which is a contradiction. This concludes the proof of Theorem 1. □

As observed by Subrahmanyam Kalyanasundaram, the proof of Lemma 2 can be transformed into a polynomial time algorithm which outputs two L-colorings f_1, f_2 and a color c such that the weight of c differs in f_1 and f_2, assuming the total weight is not divisible by $d + 1$. This implies that the coloring of Theorem 1 can be found in polynomial time.

AI disclosure. ChatGPT Sol 5.6 found the paper of Schauz [Sch10] giving a combinatorial proof of the Alon-Tarsi theorem and suggested the proof of Theorem 1 given in this paper. No AI tools were used in the writing of this note. I take full responsability for its content, but of course do not claim any of the original ideas in it as being my own.

References

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